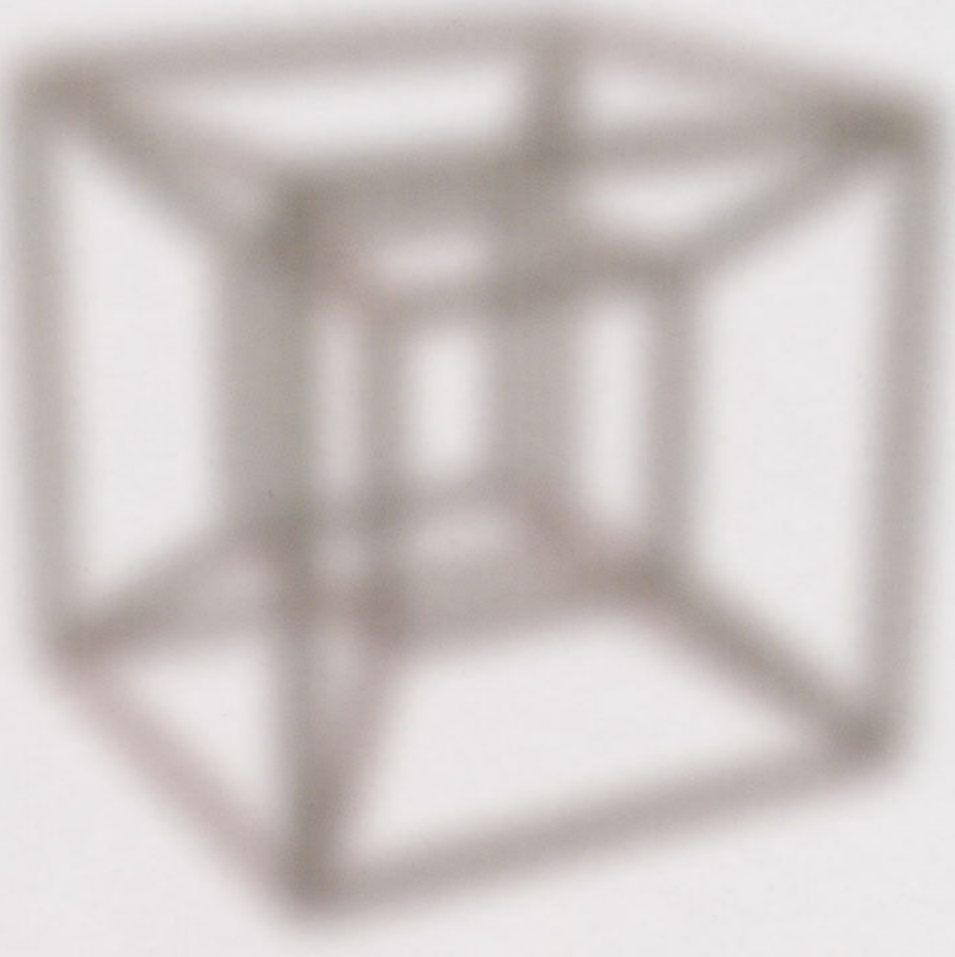
The Fourth Dimension

Is our Universe the shadow of another?



Everything is mathematical

The Fourth Dimension

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Is our Universe the shadow of another?

Raúl Ibáñez

Everything is mathematical

For my mother, a strong and positive woman.

For my wife Ana, and my children Aitor and Vanessa

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Preface

It is not often that mathematical theories descend from the ivory towers of academia to mingle with mainstream culture. When this does happen, they are often timid and sporadic incursions or superficial passing trends. However, at the end of the 19th century and the start of the 20th, society became fascinated by the possibility of the existence of dimensions beyond our own three-dimensional reality.

Under normal circumstances, the double revolution marked by the discovery of non-Euclidean geometries and the birth of multidimensional differential geometry would have passed unnoticed among members of the public at the time, and would have only attracted the attention of the scientists who saw how important they were to the future of maths, science and engineering. However, the fourth dimension seems to have captured the collective imagination and was addressed in a number of popular publications.

It was the mathematicians themselves who began the craze after introducing the ideas through conferences, articles and books, first directed at the scientific community and, later on, the general public. The concept soon caught on as we will see throughout the course of this book.

Scientists made use of the fourth dimension to try to describe the universe. Multidimensional spaces became a highly useful tool. Philosophers pondered the concept of space, the shape and structure of the universe, our own existence. More generally, theologians and religious figures approached the fourth dimension as a path to formulating theories on God, heaven and hell, souls, spirits and the existence of higher realities. Mystics, spiritualists, theosophists and many alleged prophets also made use of a universe that was revealed to them by the fourth dimension. Writers employed its most fascinating aspects in their work, such as imagining how four-dimensional beings would look, taking journeys to other dimensions, and employing the miraculous powers of those who could travel to the fourth dimension in their stories about parallel universes and time travel. In the art world, it meant a break from Renaissance perspective, a new language and a path to a new, previously unseen reality. And so the extra dimension offered many avenues for the general public to travel along and many became fascinated by the possibility that we, or perhaps that which many referred to as our souls, existed in a four-dimensional space.

One of the books that made the greatest contribution to the dissemination of the fourth dimension at that time, and which continues to be a good starting point for those interested in this mathematical concept, was *Flatland: A Romance of Many Dimensions*. And our book will take off along its trajectory from the perspective furnished by this very novel.

Chapter 1

Flatland: A Romance of Many Dimensions

> Me [a square]: *But my Lord has shown me the intestines of all my countrymen in the Land of Two Dimensions by taking me with him into the Land of Three. What therefore more easy than now to take his servant on a second journey into the blessed region of the Fourth Dimension, where I shall look down with him once more upon this land of Three Dimensions...*
> Sphere: *But where is this land of Four Dimensions?*
> Me: *I know not, but doubtless my Teacher knows.*
>
> Edwin A. Abbott, *Flatland*

Flatland: A Romance of Many Dimensions is, without a shadow of doubt, the book that has made the biggest contribution to the dissemination and popularisation of the fourth dimension among mathematicians, scientists and students, as well as thinkers, artists and the general public. It was published in 1884 and has remained popular ever since. The book continues to attract a degree of interest, with new editions still being printed despite also being freely available on the Internet.

It is not so much a popular science book as a work of fiction which introduces the reader to the fascinating world of the fourth dimension (and ones beyond) by means of analogies. Its author invites us to enter the mind of a two-dimensional being and explore the flat world in which he lives in order to then confront him with the evidence that there are worlds with fewer and more dimensions than his own (one and three, respectively). This allows the reader to experience the inherent difficulty in imagining a reality with more dimensions than those revealed by their own senses. However it also shows that it is perfectly possible for these imperceptible dimensions to exist. The author proposes a thought experiment that helps us understand the fourth dimension from our own three-dimensional existence. There are people who, having read the book, are left with the impression that it is just a maths-based novel. However, *Flatland* is much more than this. It is a social satire of

O day and night, but this is wondrous strange
FLATLAND
A ROMANCE
OF MANY DIMENSIONS
By A Square
THE HALL
THE CELLAR
LONDON
SEELEY & Co., ESSEX STREET, STRAND
Price Half-a-crown
And therefore as a stranger give it welcome

Cover of the first edition of Flatland: A Romance of Many Dimensions.

the Victorian England in which the author lived, and a metaphor used to discuss the theological issues that greatly concerned him.

The central idea of the book, which brings the mathematical, social and theological strands together, is to encourage its readers to break the chains of their limited perception of reality and open their minds to new forms of perception, to new ideas, to a new world. The simplicity of both the ideas present in the story and the language used have helped this book to have a strong impact on a wide range of readers. We shouldn't forget that its author, as we shall see later on, was a Church of England vicar who was convinced that sermons should be simple so as to be understood by everyone. He was also headmaster of a school, hence his special interest in education. But the success enjoyed by *Flatland* is not only a result of the book's intrinsic qualities, but also of the growing interest in the existence of a multidimensional universe that reflected the development of non-Euclidean geometries at the time.

The author: Edwin Abbott Abbott

As we can see on the cover of Flatland, the book is authored by A Square, a pseudonym used by Abbott and perhaps a play on words based the novel's protagonist, the square who lived in Flatland, and "A squared", since the repetition of his two last names means that Abbott could have been dubbed "Abbot Squared".

Edwin A. Abbott was born in London in 1838. He was educated at the City of London School and then at St. John's College at the University of Cambridge, where he read classics, focusing his efforts on classical literature, grammar and theology. He was ordained into the Church of England in 1862 and married a year later. At only 26, he returned to the City of London School as the headmaster.

We can infer from his occupation as an educator and second role as a vicar that Abbott was a socially responsible person and also a radical thinker. He succeeded in introducing new ideas both at the school at which he was headmaster and at the meetings of the English Headmasters' Conference. Abbott believed that education was the way to break down social barriers and he fought for the rights of the poorest classes as a leading member of the Movement for Opportunities for Young Men and Women of all Social Classes.

He dedicated his intellectual life to the study of grammar, literature and theology, writing more than 40 books and numerous articles on these subjects. Among his extensive output, *Flatland* is the only work with contents related to mathematics. But how is it possible that Abbott, a man without a specific mathematical education, could become interested in the fourth dimension and be the creator of the book that brought the subject to mainstream society?

Edwin A. Abbott in 1884, the year Flatland *was published.*

Abbott's best friend, the mathematics teacher Howard Candler, with whom he maintained an extensive correspondence, taught at Uppingham School. Coincidentally, the English mathematician Charles H. Hinton, one of the principal philosophers and writers on the fourth dimension, also taught there. Abbott may have met Hinton at Uppingham or learned about the subject through his friend Candler. Whatever the case, he clearly understood the concepts of the fourth dimension well enough to be able to use them as a metaphor of the social and theological status quo in the class-ridden world of Victorian England.

The aims of the book

As we have already seen, *Flatland* is not just a sci-fi novel but at its heart is an allegory that makes use of geometrical shapes and dimensions in order to tackle the big issues of the day. Aside from the mathematical concept of dimension, Abbott's aims in writing the book were twofold: social commentary and theological reflection.

In terms of social commentary, *Flatland* is clearly a satire of the English society of the time, characterised by a rigid class system and resistance to change of any sort. Abbott rounds on the harshness with which the most needy classes were treated, being excluded from education, which was an exclusive privilege of the social elites. He also rails against the subjugation of women and the rejection of new ideas. This use of social satire followed in the footsteps of several distinguished predecessors, such as Jonathan Swift and his book *Gulliver's Travels* (1726) and Lewis Carroll's *Alice in Wonderland* (1865).

In addition to social commentary, Abbott also wanted to discuss his pet theological subjects, which he had previously addressed in a more direct manner in his other books and articles. Some passages, such as the journey of the square protagonist through higher dimensions, can be interpreted as a metaphor for mystical experiences in higher realities. Similarly, the author criticises the dependency on miracles as the basis for religious beliefs, while also trying to show that science, while being able to progress the human species by providing greater knowledge of the universe, could never bring it closer to God. Finally, we can observe a certain parallel between the square, his attempt to explain the third dimension and his treatment of the mysteries of the three dimensions, and the apostles, their evangelising work and the Bible.

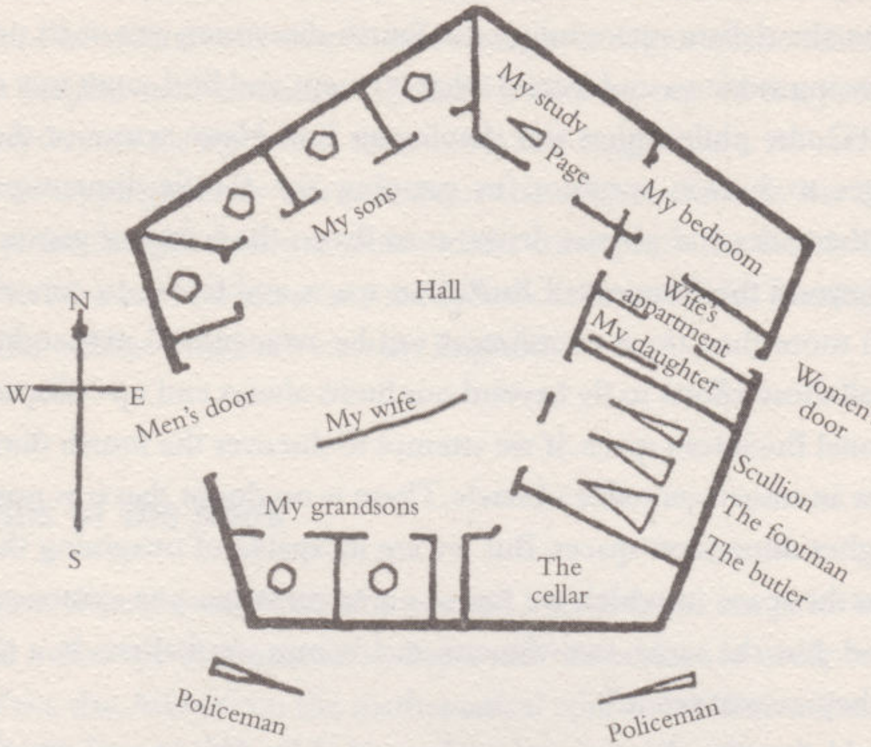
However, what makes *Flatland* a memorable work is its mathematical content. In Abbott's time, the debate surrounding the fourth dimension was at its peak and attempts were being made to understand what it meant and find some way of visualising it. In 1952, the philosopher and theologian Karl Heim wrote of the enormous challenges to human intuition in grasping the fourth dimension: "The progress of mathematics and physics drives us to fly on the wings of poetic imagination, going beyond the frontiers of Euclidean space, and trying to conceive of a space in which more than three coordinates can be arranged, all perpendicular to the other. But all those efforts to fly beyond our limits always end up collapsing into three-dimensional Euclidean space. If we attempt to discover the fourth dimension, we come across an insurmountable obstacle. There is no doubt that it is possible to calculate in higher dimension spaces. But we are incapable of imagining them. We are confined to the space in which we found ourselves when our existence began, as if imprisoned. Just the same, two-dimensional beings can believe in a third dimension, but they cannot see it."

It can be said that the dimensional analogy used by Abbott and which was a fundamental tool of that time, brought us closer to how it was possible to 'see' the invisible.

Part one: This world

Flatland is written in the first person by a square, a mathematician by profession, who tells us about the strange adventure that he has lived through. As a result he learns more about the reality of the universe but is also imprisoned in a cell from which he writes his story. As such, the first part of the book gives a description of his world, Flatland, which is two dimensional, and the society in which he lives. It is this part that contains most of the social commentary.

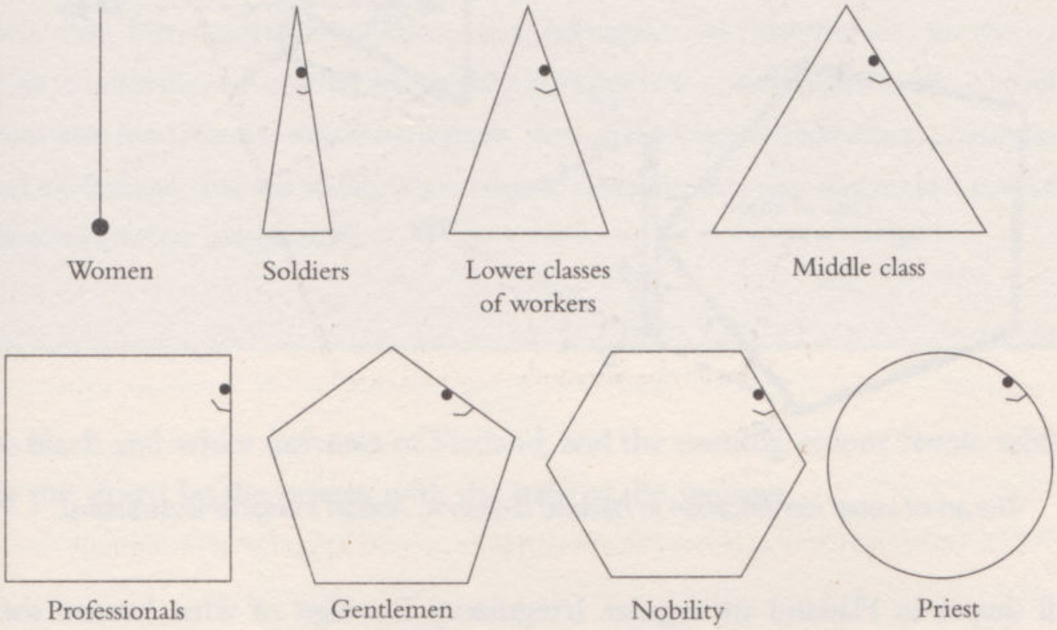
As we have said, the universe of the square protagonist is flat, two dimensional ("imagine a vast sheet of paper", Abbott explains), inhabited by straight lines, squares, pentagons, hexagons and other polygonal shapes. With the exception of fortifications, barracks and official buildings, the houses in which the shapes live are often pentagons, with the roof pointing towards the north, since there is a constant attraction towards the south, meaning that the rain always 'falls' from north to south. In addition to this, the houses have two doors, one for men and another for women.



Typical pentagonal house in Flatland (Edwin A. Abbott's original illustration).

Next Abbott describes the inhabitants of this curious world. Women are straight lines; soldiers and the lowest classes of workers are isosceles triangles; the middle class is made up of equilateral triangles, and professionals and gentlemen are squares and pentagons. Directly above these is the nobility, which are hexagonal, and from then on, the number of sides gradually increases with social standing. Finally – the number of sides is so large that the figure cannot be distinguished from a circle – there are the priests. The internal angle of the figures (smallest in the isosceles triangles), is obviously related to the number of sides, and reflects the social position and education of the figure. In addition to this, male children have one more side than their fathers, although this does not always happen among merchants and less still among soldiers and the lowest classes of workers. When, by some miracle, the son of an isosceles triangle is equilateral, he is separated from his parents and adopted by a childless equilateral.

Women are straight lines without angles, without education, and without rights in the society. We can see this in one of the paragraphs of the book, where Abbot writes: "Not that it must be for a moment supposed that our women are destitute of affection. But unfortunately the passion of the moment predominates in the frail sex over every other consideration. This is, of course, a necessity arising from their unfortunate conformation. For as they have no pretensions to an angle, being inferior in this respect to the very lowest of the isosceles, they are consequently wholly devoid of brain power, and have neither reflection, judgment, nor forethought, and hardly any memory. Hence, in their fits of fury, they remember no claims and recognise no distinctions."

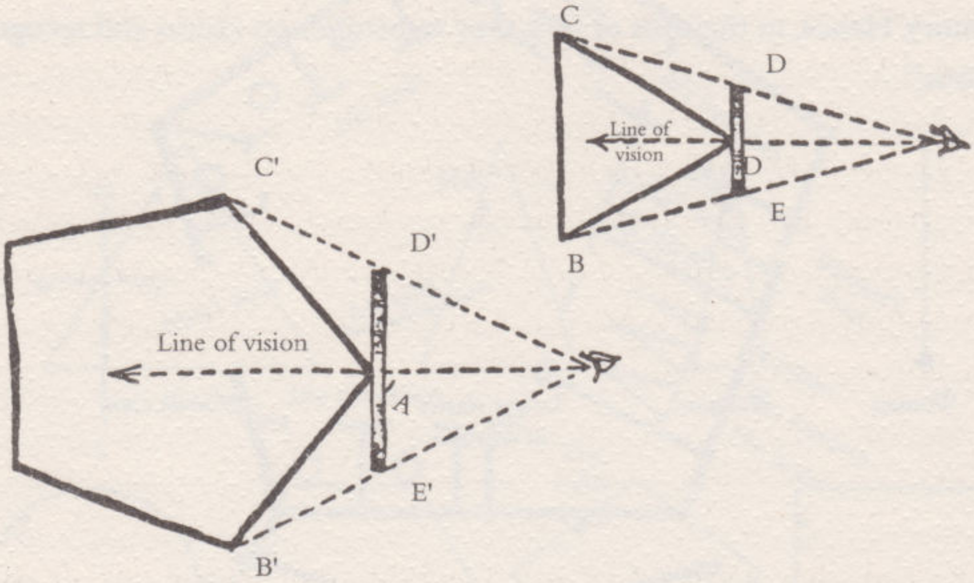


The range of geometrical shapes distinguish the social classes of the inhabitants of Flatland.

In this society, men, especially those belonging to the upper classes, attempt to justify the lack of women's rights and social exclusion by claiming that they are not a result of discrimination by society itself, but are a product of their nature, the configuration of their dimensions.

Inhabitants of Flatland recognise each other in different ways. The lower classes and women do so by touch. Equilaterals, squares and pentagons use their hearing, identifying the voice of their visitors. The upper classes make use of visual identification. Any inhabitant of Flatland appears to them as a straight line, however the constant fog that covers in this world, allows them to make out depth and thus a character's angles. The fog means that visibility decreases with distance and as such, when an angle is small, such as the angle of an isosceles triangle, its sides begin to fade almost immediately, while for a larger angle, this occurs more slowly. Recognition by touch is taught at schools, chiefly by means of 'hands-on' practice. Isosceles triangles between half a degree and ten degrees are used in lessons. These individuals do not having sufficient intelligence for war and so are used as educational props.

The science and art of visual identification is taught to the social elites at universities, and requires the study of geometry.



The art of visual identification in Flatland (Edwin A. Abbott's original illustrations).

All shapes in Flatland are regular. Irregularity is a sign of what human society would recognise as perversity and delinquency. We can see this in the description by the square: "The irregular, they say, is from his birth disowned by his own parents, derided by his brothers and sisters, neglected by domestics, scorned and suspected by society, and excluded from all posts of responsibility, trust, and useful activity. His every movement is jealously watched by the police till he comes of age and presents himself for inspection. Then he is either destroyed if he is found to exceed the fixed margin of deviation, or else immured in a government office as a clerk of the seventh class, prevented from marriage; forced to drudge at an uninteresting occupation for a miserable stipend; obliged to live and board at the office, and to take even his holidays under close supervision…".

At the other end of the social scale are the priests. "Our priests are administrators of all business, art, and science; directors of trade, commerce, generalship, architecture, engineering, education, statesmanship, legislature, morality, theology. Doing nothing themselves, they are the causes of everything worth doing that is done by others." Their doctrine is that they must worry about configurations, since this determines the roles of each shape.

The rejection of new ideas and anything that would mean breaking the established social order can be seen in the chapters that deal with the inclusion of colour

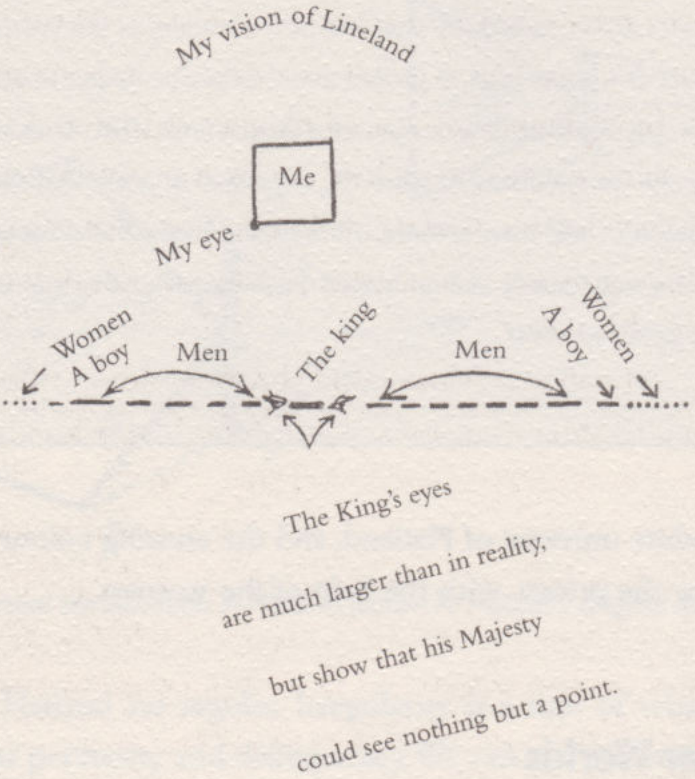
CHAUVINISM IN FLATLAND

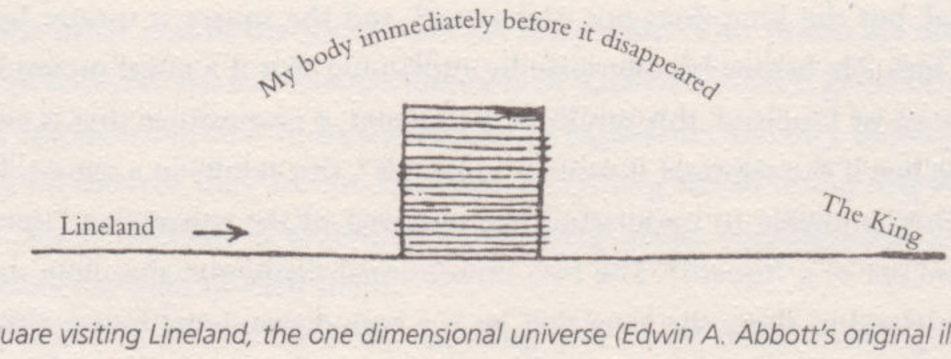
Upon reading *Flatland* for the first time, there are people who take against the book, branding its author a chauvinist. However this could not be further from the truth: Abbott was a strong advocate of women's rights. In fact, he was one of the leaders of the Women's Education Movement. In 1870, the universities of Oxford and Cambridge began to admit women to study there, even if they were not able to earn a full degree until 1920. Women were allowed to go to university and were required to reach the same level of atainment as men, although there were few schools to help them prepare. Through the English Headmasters' Conference and the Teachers' Training Society, Abbott helped create opportunities that made it possible for girls to receive suitable tuition.

in the black and white universe of Flatland, and the ensuing colour revolt, which is finally put down by the priests, with the help of the women.

Part two: Other Worlds

The second part of the book, entitled *Other Worlds*, is more focused on the problems of dimensional analogy and theological aspects, although the social commentary continues until the end of the book. First of all, the square, in a strange dream, visits Lineland, a universe that is made up of an infinite straight line, and, as such, is one-dimensional, inhabited by segments (men) and points (women). There, from outside Lineland, he speaks to the king of this world, who is unable to understand with whom or what he is speaking. Our protagonist attempts to explain to the king that he lives in a two-dimensional world and that his own existence is two dimensional, but the king does not understand, and the square is unsure how to explain to him. He begins his discourse by explaining that if a point moves in the only direction of Lineland, this results in a segment, a proposition that is evident to the king, but if the segment is moved "upwards", this results in a square. However, the king is unable to understand the meaning of the expression "upward", nor the concept of a "square". The two-dimensional mathematician then decides to cross Lineland to show the king that he is a two-dimensional being, although the sovereign does not believe that he is seeing the different sections of a square, but thinks it is a segment of Lineland, which is able – in some incomprehensible way – to appear and disappear.

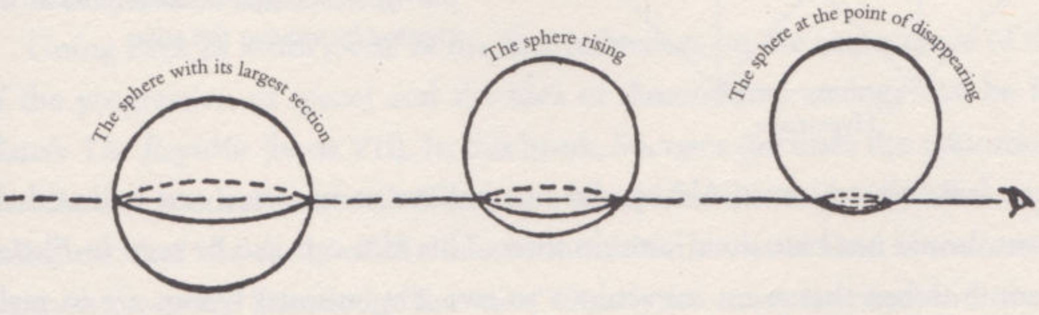

My vision of Lineland
Me
My eye
Women
A boy
Men
The King
Men
A boy
Women
The King's eyes
are much larger than in reality,
but show that his Majesty
could see nothing but a point.



The square visiting Lineland, the one dimensional universe (Edwin A. Abbott's original illustrations).

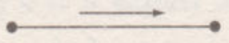
Upon waking the following day, the square receives a mysterious visit from a sphere, who lives in Spaceland, the three-dimensional universe that contains Flatland. To start with, just as happened to the king of Lineland, our protagonist is unable to decipher where the voice is coming from. This time it is the sphere who is trying to describe the three-dimensional nature of space to the inhabitant of Flatland, arguing that if a square figure grows in a new direction 'upward', a cube is obtained, which has three dimensions. Faced with the failure of the pupil to understand the argument of his teacher, the sphere decides to cross Flatland so that its flat sections, which are circular, can be seen, although the square thinks it is a priest who appears magically, grows as if time were passing faster than normal, then mysteriously shrinks and disappears.

Continuing the dimensional analogy and the study of social mobility with respect to dimensions, the three-dimensional visitor decides to attempt an argument based on the number of vertices (corners) and faces. The vertices of a point, a segment and a square give the geometric progression, 1, 2, 4, which continues with the number 8, which, as the sphere explains to the square, is the number of vertices of a cube. Similarly, a point does not have faces, a segment has two (its two end points) and a square has four (its four sides). This gives the arithmetic progression, 0, 2, 4, which continues with 6, the number of faces of a cube.

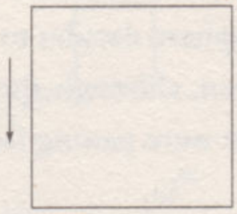

The sphere with its largest section
The sphere rising
The sphere at the point of disappearing

The illustration shows the sphere visiting the square passing through Flatland.

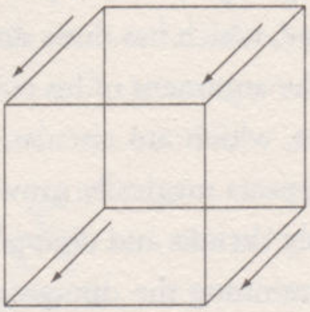
The visitor, powerless in the face of his pupil's failure to understand, decides to take drastic measures and removes him from Flatland; this is possible thanks to the fact that Flatland, as well as its inhabitants, has a fixed thickness in three-dimensional space. Seeing his world from outside, he understands the meaning of the third dimension of space to which his teacher alluded. Suddenly, all the arguments he has been given become clear, but that is not all. As a good mathematician, he realises that such arguments allow him to go further. After reflecting for some time, he remarks to the sphere that if the same dimensional analogy is used, there is perhaps also a four-dimensional space upon which the universe of the sphere rests. It is now the sphere that is confused, refusing to admit the square's argument for the existence of a four-dimensional space: "There is no such land. The very idea of it is utterly inconceivable."



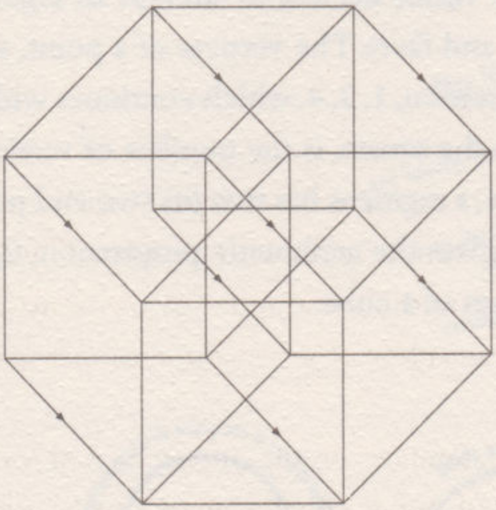
Segment



Square



Cube



Hypercube

If a point (zero dimension) is moved in a certain direction, a segment is created (first dimension). If this is moved in a perpendicular direction, a square is created (second dimension). By moving the square in a perpendicular direction, a cube is created (third dimension), and the hypercube (fourth dimension) is created by moving the cube.

As we have already seen, Abbott did not believe in miracles, and believed that Christians should not base their faith on them. This idea can also be seen in *Flatland*, where some things that seem miraculous to two-dimensional beings are in reality easily explained by passing into the third dimension. Here we reproduce a somewhat ironic dialogue in this respect, between the square and the circle:

> Awestruck at the sight of the mysteries of the earth, thus unveiled before my unworthy eye, I said to my Companion,
>
> "Behold, I am become as a God. For the wise men in our country say that to see all things, or as they express it, *omnividence*, is the attribute of God alone." There was something of scorn in the voice of my Teacher as he made answer: "Is it so indeed? Then the very pickpockets and cut-throats of my

country are to be worshipped by your wise men as being gods. For there is not one of them that does not see as much as you see now. But trust me, your wise men are wrong."

The novel ends with its protagonist imprisoned for trying to explain the existence of three-dimensional space to the inhabitants of his flat universe and for writing a treatise on the mystery of the third dimension. Here, for example, we can see an analogy between the scriptures and the persecution of the apostles. We can even see how Abbott's language takes on a biblical tone in this part, such as when he says, in reference to the square: "Death or imprisonment awaits the Apostle of the Gospel of Three Dimensions". The social commentary is also kept up, portraying a society that punishes those who attempt to introduce new ideas.

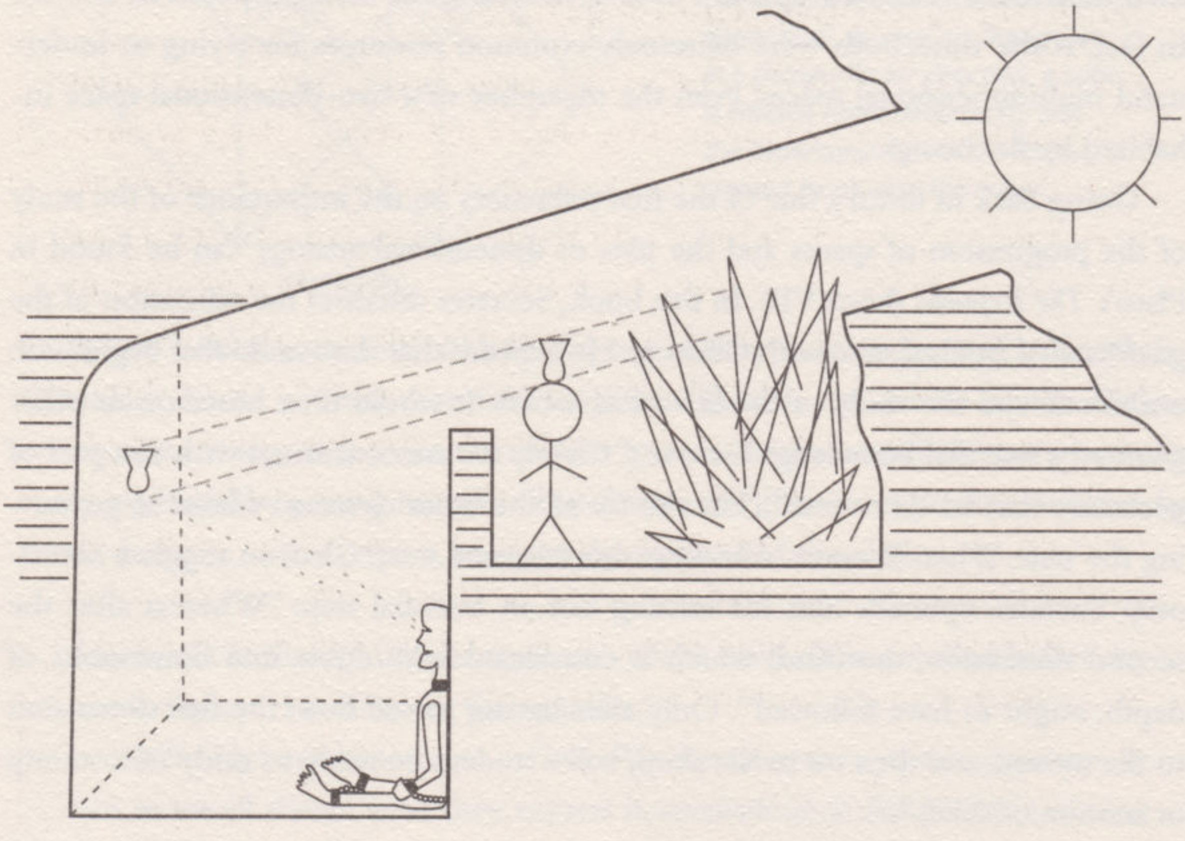
The context of *Flatland*

The dimensional analogy and the study of the progression of spaces with respect to their dimensions, key concepts in *Flatland*, were original ideas proposed by Abbott. In fact, at the time, both were extremely common resources for trying to understand multidimensional spaces, even the metaphor of a two-dimensional space inhabited by flat beings.

Going back in history, one of the first references on the importance of the study of the progression of spaces and the idea of dimensional analogy can be found in Plato's *The Republic* (book VII). In this book, Socrates discusses the education of the guardians of an ideal state with Glaucon. He explains that this would first begin with arithmetic and the study of the line of numbers. It would then move on to plane geometry, essential knowledge for war ("Clearly we are concerned with that part of geometry that relates to war"), but also for all the other elements related to governing the state. When Socrates asks what should come next, Glaucon suggests astronomy. Socrates upbraids him for missing out an essential step: "Whereas after the second dimension, the third, which is concerned with cubes and dimensions of depth, ought to have followed". Only after having passed from the first dimension to the second, and then on to the third, will a student be ready to study "astronomy, or motion of solids".

The celebrated myth of the cave, also the work of Plato, is another fundamental reference point for the issues dealt with in *Flatland*. Here we also find dimensional analogy, the problem of knowing the world in which we live and education as a means to achieve this knowledge. Plato asks us to imagine a race of men who are trapped in a dark, underground cave from birth, bound in such a way (body, legs, hands, neck) that all they can see is the wall of the cave in front of them. Behind the men is a small wall with a fire behind it. Between the fire and the wall are objects in the shapes of small men, animals and tools, which are moved from side to side, in such a way that the fire projects the shadows of these objects onto the wall of the cave. When the prisoners speak, their voices bounce off the wall and they think that the shadows are speaking, and even that they themselves are shadows. The cave dwellers think that these shadows are the only reality and do not understand that their own existence and that of the objects is located in three-dimensional spaces. It is interesting to point out the end of the story, in which someone from outside visits them and tries to explain to them the true reality of the world, but they dismiss him as a madman.

A diagrammatic representation of the myth of Plato's cave.



Another relationship between the myth of the cave and the fourth dimension is the fact that the prisoners think they are two-dimensional beings. Their condition as three-dimensional beings is as strange to us as if we were really three-dimensional projections or sections of four-dimensional beings.

In the middle of the 19th century, we can already find an idea similar to the myth of the cave in the short story *Space Has Four Dimensions*, by the German psychologist and physicist Gustav Fechner (1801-1887), in which the man-shadow is projected onto a vertical screen by a dull projector.

Other accounts of flat worlds

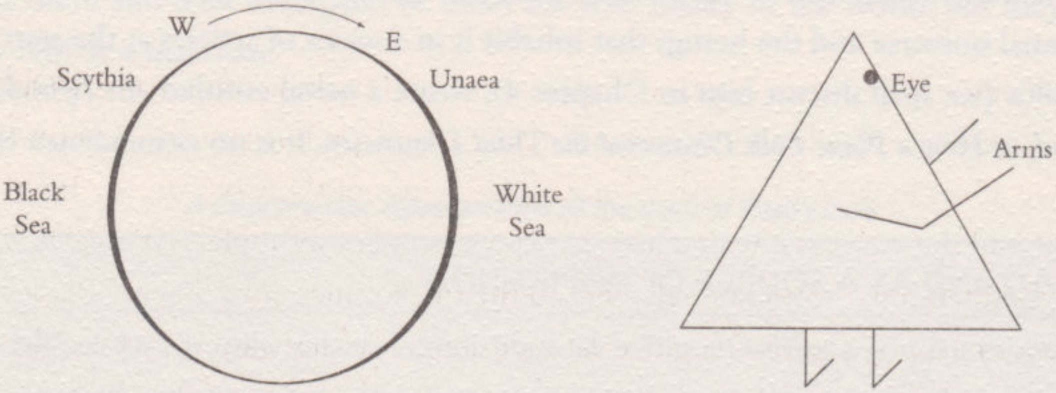
The mathematician Charles H. Hinton, who had already written about a two-dimensional universe and the beings that inhabit it in a series of articles at the start of the 1880s (we shall discuss him in Chapter 4), wrote a novel entitled *An Episode of Flatland, or How a Plane Folk Discovered the Third Dimension*. It is no coincidence that

FLATLAND AS A SOURCE OF INSPIRATION

Flatland's status as a popular classic has led many authors to write works directly inspired by it. *Sphereland, A Fantasy About Curved Spaces and an Expanding Universe*, by Dionys Burger (1892-1987), was devised as a continuation of the novel and was written with the same relatively simple set up. The protagonist is a hexagon, the grandson of the square, in a more egalitarian society. Upon measuring a large triangle cast against the two-dimensional universe, it is discovered that its angles measure more than 180°, leading its inhabitants to think that they really live in a spherical space, not a flat one. Not even Ian Stewart (1945-), one of the most famous exponents of popular mathematics in our times, has been able to resist the temptation to visit *Flatland*, producing an annotated version and even a sequel, *Flatterland, Like Flatland Only More So*. The protagonist of the book, Victoria Lane, a descendent of the square in Abbott's classic, takes a tour of more modern concepts, such as the fractal dimension, hidden spatial dimensions, hyperbolic geometry, quantum mechanics, the theory of relativity, space-time singularities and time travel.

Hinton and Abbott's books were written at around almost the same time.

In Hinton's flat universe, there are circular planets orbiting a circular sun. One of these planets is Astria, the inhabitants of which are triangular. There are two races that live in separate areas, the civilised Unaeans, who have developed science and technology, and the barbarous Scythians, who are warriors. In this story, Hinton explores, to a greater extent than Abbott does in *Flatland*, the debate on science and technology in Astria. In particular, he speculates on the physics of that two-dimensional world and some mechanical devices. And of course, nor is the work lacking a social angle, in this case the relationship between a young bourgeoise lady and a member of the proletariat. Her uncle is the only person on Astria who believes in the existence of three-dimensional space.



Illustrations based on the book An Episode of Flatland, *by C.H. Hinton. The story takes place on the planet of Astria, circular and flat, whose inhabitants are triangular. In the west live the Scythians and in the east the Unaeans.*

FROM FLATLAND TO PLANIVERSE

Using computers to represent *Flatland* gave rise to *The Planiverse, Computer Contact with a Two-Dimensional World* by A.K. Dewdney in 1984. The mathematician and scientist, who was born in Canada in 1941, studied all sorts of issues in a two-dimensional world similar to that described by Hinton. Everything from politics, geography, architecture, physics, chemistry, biology, culture and even games are explored in detail, even pondering questions such as what and how do the inhabitants of this world eat?

Chapter 2

What is a dimension?

> *I know that there are many … who consider the supposed notion of generalised* [four-dimensional] *space as nothing more than a masked form of algebraic formulation; but the same could be said of our idea of infinity in algebra, or of impossible lines, or lines that form an angle of 0 in geometry, and nobody will argue the usefulness of working with them.*
>
> *James J. Sylvester,* A Plea for the Mathematician *(1869)*

This chapter explains the concepts of dimensions and multidimensional spaces. The term *dimension* is widely used in science and technology but also in our everyday language. If we look at a newspaper or search for uses of the word on the Internet, we will find the word in a wide range of expressions. For example, "GPS navigation in three dimensions" would be referring to the three dimensions that a GPS device uses to locate a position on the globe: latitude, longitude and altitude, while "the dimensions of the box are 30 cm (length) × 15 cm (width) × 15 cm (height)" is using a formula that is used to express dimensions of different magnitudes. We may even find something like "the cultural dimension of the Internet", which we could interpret as using the term metaphorically to illustrate the many facets of the Internet and our culture as a whole.

The use of the word *dimension* in our everyday life is essentially the same as the way it was employed in the scientific world until the 19th century, and ever since its scientific conception the meaning has been evolving through the popularisation of the original mathematical ideas. Even when used in less-than-specific phrases, such as "living in another dimension" or "journey to another dimension", its meaning is still based on the same fundamental ideas. Also, in science and technology the term has acquired several different meanings, and even a certain theoretical complexity, depending on the area in which we are working, such as dimensions of a vectorial space, topological dimensions, fractal dimensions... However, it is not the intention of this book to go into these technical aspects, but to offer a more intuitive understanding of dimensions.

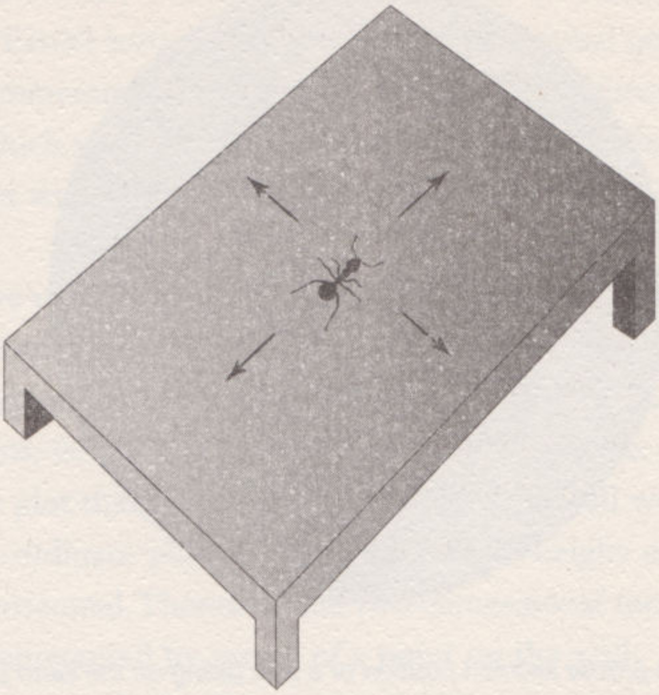
Degrees of freedom

Firstly let's focus our attention on the question: "What is a dimension?" In general, when we talk about the dimensions of a space we are referring to what physicists and engineers call "degrees of freedom".

A one-dimensional space is one in which we only have one degree of freedom, that is to say, we can only move back and forth along one path. When we travel by train, it always moves forwards on the rails – or perhaps backwards – but it cannot make any other movement. The rails it moves on describe a curve – albeit a very irregular one – but a curve in one-dimensional space. If we were in a meadow observing the paths along which ants move, we would see that they are also curved. The insects walk along them taking food to the ant hill or going back for more supplies. Likewise, the only movement possible for the king and the other inhabitants of Lineland is forwards and backwards.

In simplified terms the routes taken by foraging ants are one-dimensional spaces, as the insects move in lines, travelling back and forth in one direction or the opposite.

Ants tend to walk in lines because they follow the scent of pheromones left by those preceding them. However, pioneer ants, those that find the way, move in all directions. If we release an ant on the surface of a table we can see that it goes backwards and forwards as well as right and left or any other direction in between. A table is a two-dimensional space, that its to say, it has two degrees of freedom.

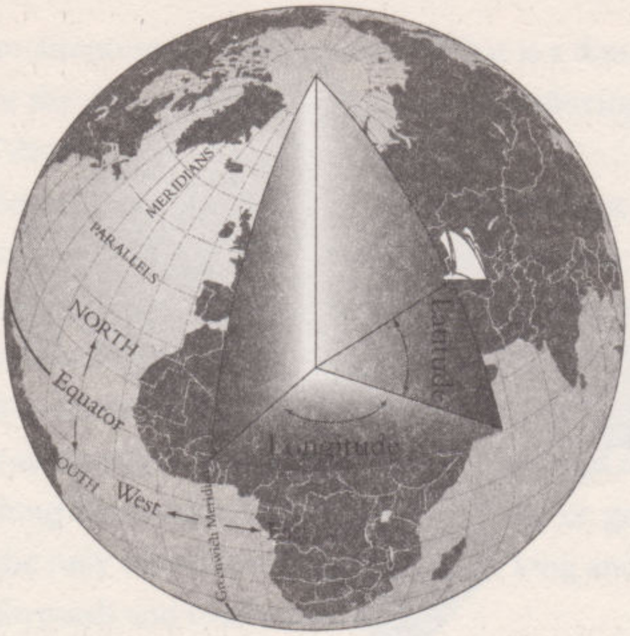


A pioneering ant on a table will move with two degrees of freedom, as it goes forwards, backwards and from side to side.

This ant has the same freedom of movement as a square living in Flatland. A ship sails on the surface of the sea and a mountaineer scales a mountainside, both two-dimensional spaces. The position of the ship or of the mountaineer on the surface of the globe can be determined using two parameters, latitude and longitude. Equally, the position of the ant on the table can be established from its distance from both sides of the table.

If in place of the ship we consider a submarine, we add the possibility of moving up and down to specific depths, in the same way that a helicopter can adjust its height in the air, therefore having three degrees of freedom. We have now arrived at the three-dimensional space that we all experience naturally.

If the helicopter, a traffic helicopter for example, flies at certain times every day, we can add time as a new degree of freedom (although here we can only move forward; one of the peculiarities of the dimension of time – or at least our perception of it). Our lives thus run their course in four-dimensional space-time and as such could be plotted as coordinates.



We can determine the precise location position of a ship sailing on the Earth's oceans by means of two numbers, its latitude and longitude.

Coordinates

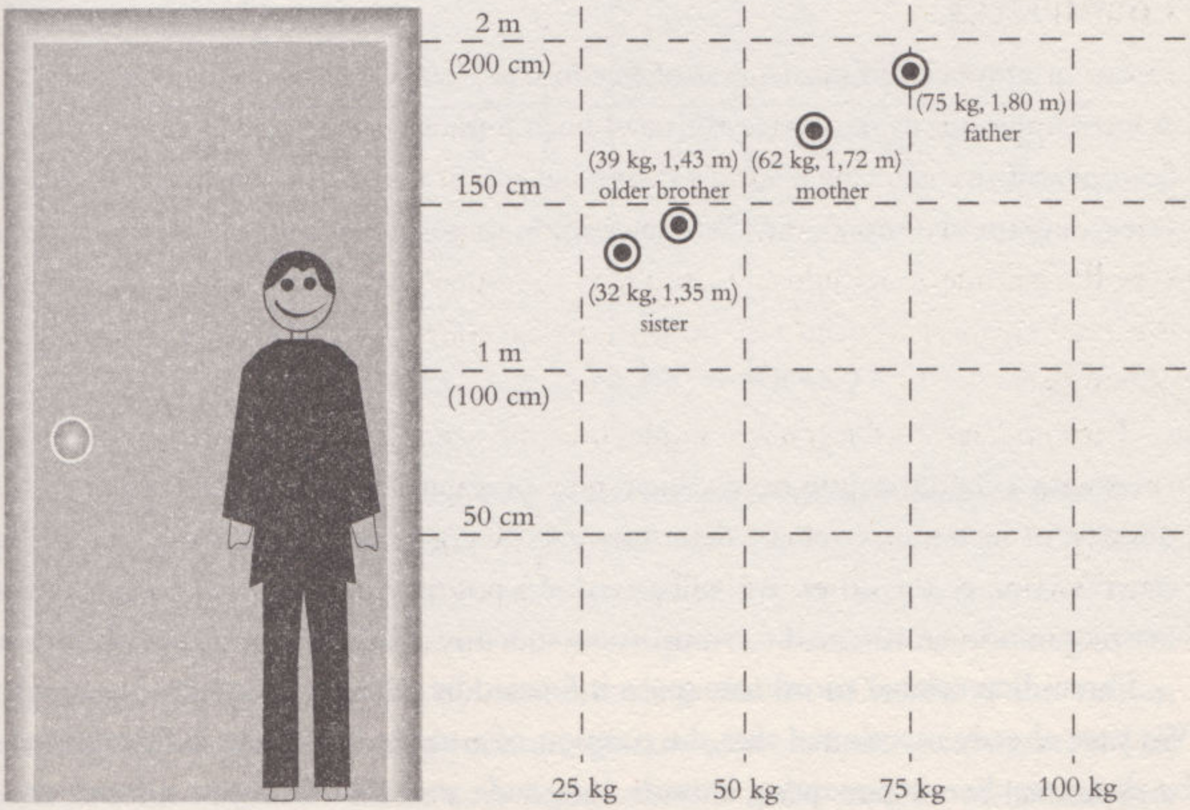
In our brief reflection on degrees of freedom we were able to see that in order to determine positions in a space we need both numeric values and degrees of freedom of that space. In the example of the helicopter moving in the sky (which is a three-dimensional space), the GPS determines its position by means of three numbers, the latitude, longitude and altitude with respect to sea level, which introduces the mathematical concept of dimensions as coordinates, in other words, lists of numbers.

Let's go back to the train example. Imagine a single railway track joining two cities with a central station from which the movements of trains are monitored. The position of each train could be determined knowing its distance from the station in one direction or the other (we will give one a positive sign and the other negative in order to distinguish the directions). Consequently, a unique number, a coordinate (x_1), is all that is needed to determine the position of the train. The space for all possible positions of the train, the track, can be identified with the one-dimensional coordinate space described by all the possible values of coordinate x_1.

Similarly, by means of one number, we express the height of the members of our family, which in some homes can be seen on a door frame, which then becomes a graphical representation of the one-dimensional space of possible heights.

With two numbers (x_1 = longitude, x_2 = latitude) we can describe the position of a place on the Earth's surface, which is a two-dimensional space. A more abstract example of two-dimensional space would be the "space" formed by hypothetical picture frames, which are determined by two measurements, the length and height, so the single point with coordinates (29, 35) determines a frame that is 29 cm long and 35 cm high.

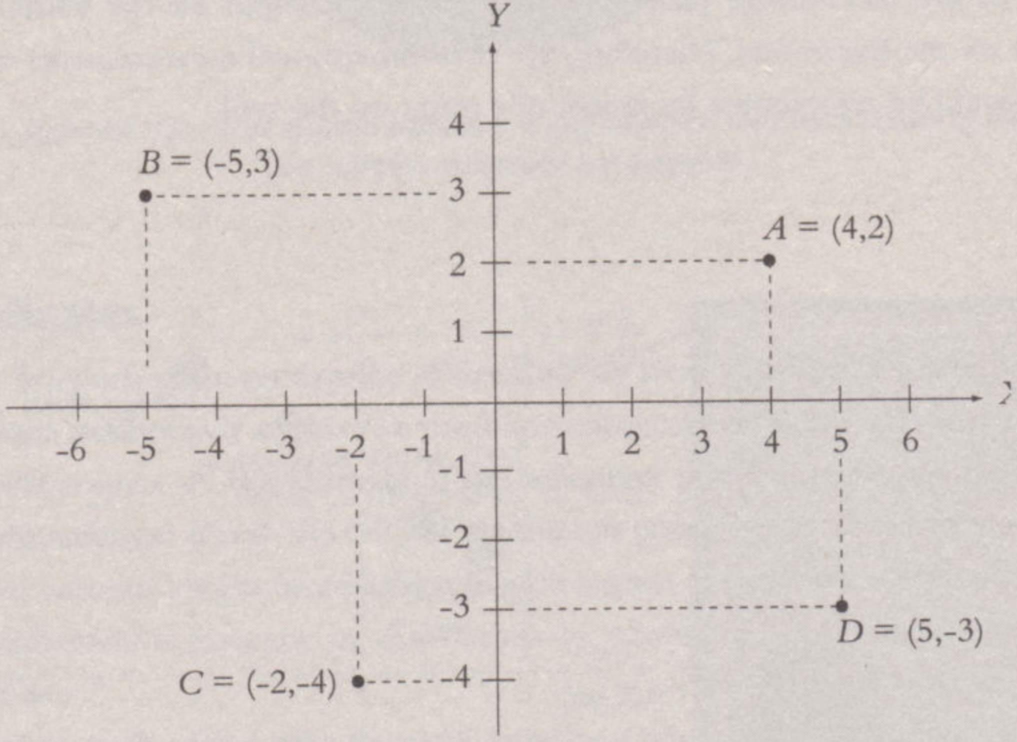
Likewise, if we reflect on the height and weight of the members of our family, these measurements are also points in a two-dimensional space bound by all the possible measurements. However, the doorpost will not be sufficient to represent these points, we would need an entire wall, which is perhaps the reason why few families plot their development like this! The wall would be a representation of the co-ordinate plane. We would express height on the vertical, and weight on the horizontal. Therefore, the two-dimensional measurements of each of us would be represented by means of a point on the wall.



The wall of a kitchen symbolises a coordinate plane, the door frame at the side being the height axis, the floor the weight axis and the four points represent the defining measurements for the family members.

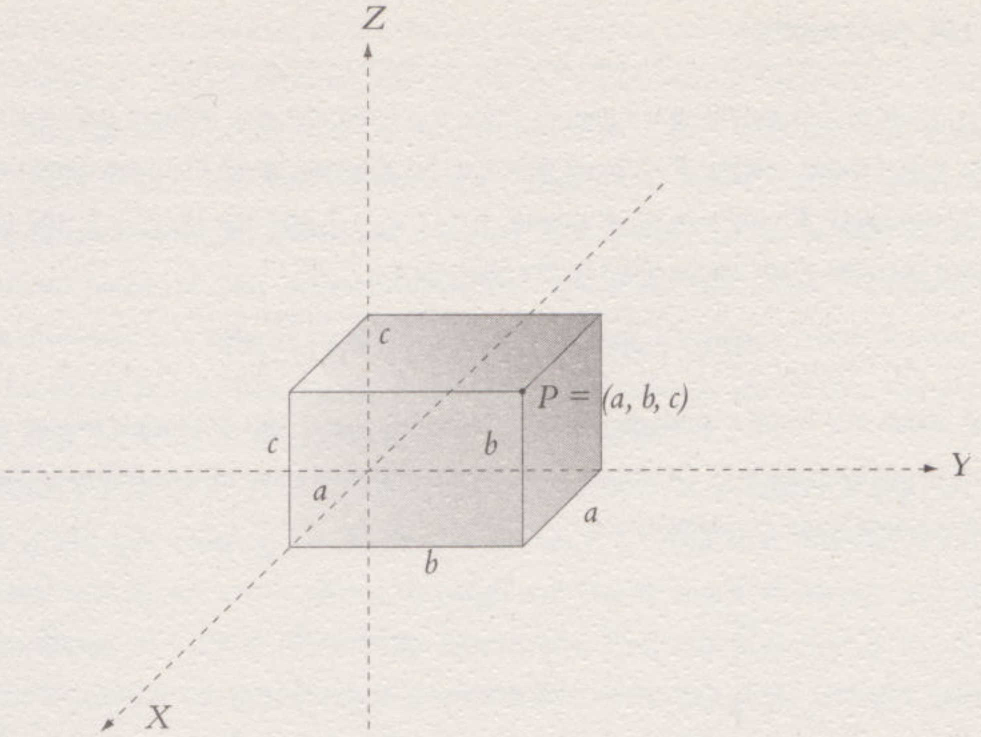
DESCARTES' FLY

French mathematician René Descartes (1596-1650) introduced the concept of coordinate plane, as well as analytical geometry, in his work *Geometry*, published as an appendix to his book *Discourse on Method*. According to an apochryphal story, the idea of the Cartesian plane came to him while he was thinking about the movement of a fly on the ceiling of his bedroom. He realised that the position of the fly could be described by its distances from the two walls. So Descartes was adding coordinates, a tool from algebra, to Euclid's plan, which in turn can be considered as being set out on a blank geometric plan. Although today coordinates seem a simple concept, in its time it was very difficult to digest, even for Isaac Newton (1643-1727), whose biography tells of the difficulty he had reading Descartes' work.



Coordinate plane, showing the points A = (4,2), B = (–5,3), C = (–2,–4) and D = (5,–3).

Three-dimensional coordinate space is formed by triples of numbers (x_1, x_2, x_3). We have already mentioned that the position of a traffic helicopter is determined by three numbers representing latitude, longitude and altitude. Again, a more abstract example would be the space contained within cardboard boxes, determined by their length, width and height.



A box shown in three-dimensional coordinate space. Coordinate point (a, b, c) *determines a shape of length* a, *width* b *and height* c.

In general, a coordinate space of n dimensions is formed by n-tuples of numbers, $(x_1,\dots,x_n)$, in other words, by lists of n numbers, whatever the dimension n. So each point of the space is an n-tuple $(x_1,\dots,x_n)$, and the n-dimensional coordinate space is the resulting combination of all the n-tuples. In mathematical terms,

$$\mathbb{R}^n = \{(x_1,\dots,x_n) : x_1,\dots,x_n \in \mathbb{R}\}.$$

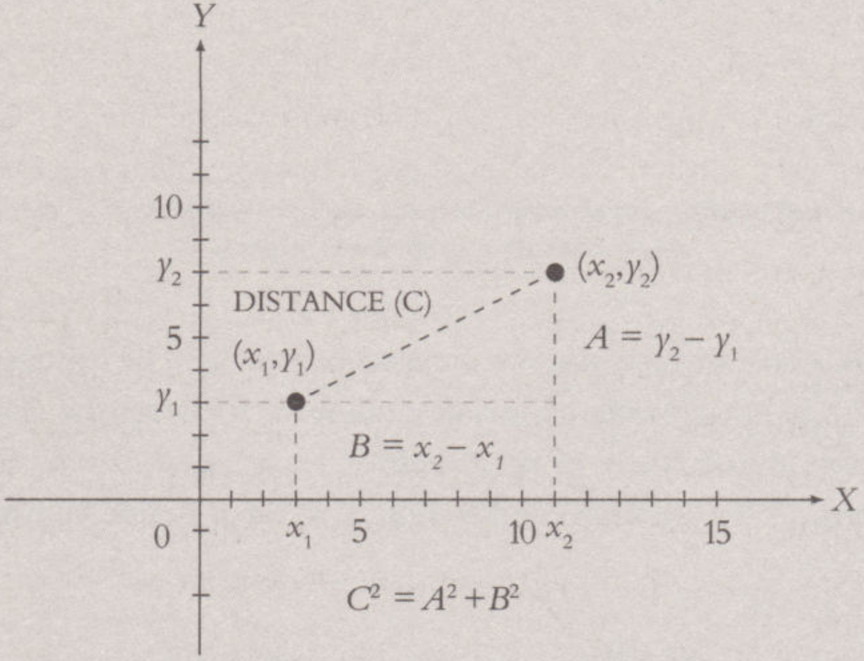
In many branches of science and technology, and even outside of them, data is provided in lists of varying numeric data, and applying the concept of coordinate space to the total number of the possible lists means that we can use geometric tools and mathematics to organise, locate and interrogate the information, yielding useful conclusions. Examples are highly varied, such as the list of results of the various medical blood tests (sodium, potassium, glucose and cholesterol levels, etc.). The complete list of results will be an n-tuple, where n is the number of clinical tests carried out. Other examples could include a student's class notes, a qualification table for a sports competition, the list of lists goes on.

USUAL DISTANCE

The concept of coordinate space implies there is a fixed distance defined between two points in that space, the so-called 'usual distance'. For example, given the three-dimensional coordinate space $\mathbb{R}^3$ and two of its points, $p=(x_1,x_2,x_3)$ and $q=(y_1,y_2,y_3)$, the usual distance between them will be given by the expression

$$d(p,q)=\sqrt{(y_1-x_1)^2+(y_2-x_2)^2+(y_3-x_3)^2},$$

which makes our world a three-dimensional Euclidean space. This distance is exactly what we use in our everyday lives. Of course, this concept of distance can be naturally extended to any coordinate space of *n* dimensions.



Distance (C) between two points (x_1,y_1) *and* (x_2,y_2) *of the plane is given by the theorem of Pythagoras, hence C is the hypotenuse of the right-angle triangle with sides* $A = y_2 - y_1$ *and* $B = x_2 - x_1$.

Are there spaces with higher dimensions?

Despite the apparent simplicity of these ideas, the truth is that it took a long time to come up with them and put them into practice, starting with the development of Greek mathematics and continuing until well into the 19th century. Mathematicians and other scientists and philosophers engaged in heated debates on the meaning and reality of higher dimensions. In Euclid's *Elements of Geometry* for example, he only talks about a point that has no dimensions, a line that has one dimension (length), a plane that has two dimensions (length and width), and a solid that has three dimensions (length, width and height). But already, in his work *On the Heavens*, Aristotle had said that four-dimensional spaces do not exist: "A magnitude if divisible one way is a line, if two ways a surface, and if three a body. Beyond these there is no other magnitude, because the three dimensions are all that there are, and that which is divisible in *three directions* is *divisible in all.*

Claudius Ptolemy (c. 100–170 A.D.) was the first to demonstrate that the fourth dimension did not exist in his work *On Distance*, a book that has not survived to this day, but which we know about through the Greek mathematician and philosopher Simplicius of Cilicia (490-560). Essentially, Ptolemy says that if we consider three perpendicular straight lines, it is impossible to draw a third straight line that is perpendicular to the other three. So the fourth dimension does not exist. Essentially however, all that Ptolemy proves is that it is not possible to perceive the existence of four dimensions within our visible space.

Later, when trying to give geometric meanings to algebraic equations, the idea that spaces of higher dimensions could exist within mathematics came up, but several voices described that possibility as 'unnatural'. English mathematician John Wallis (1616-1703) in his work *Algebra* called the fourth dimension "a monster in nature less likely than a chimera or a centaur. Length, width and thickness fill space entirely. Not even fantasy can imagine how a fourth dimension could exist beyond those three."

There were those who tried to accept a more spiritual existence of the fourth dimension. In this sense, English philosopher Henry More (1614-1687) affirmed that the spirits have four dimensions (an idea which, as we will see in chapter 5, would become very popular), and, in this regard, German Philosopher Immanuel Kant (1724-1804) wrote: "A science of all these possible kinds of space would undoubtedly be the highest enterprise which a finite understanding could undertake in the field of geometry... If it is possible that there could be regions with other dimensions, it is very likely that God has somewhere brought them into being. Accordingly such spaces would not belong to our world but must form separate worlds."

In one of his works, Kant argues that the left hand is a mirror image of the right, and that we cannot perfectly superimpose one on top of the other. However, it was

August Ferdinand Möbius (1790-1868), who observed for the first time that when rotating the right hand in a hypothetical four-dimensional space, it could become its mirror image – a left-handed version – once returned to three-dimensional space.

Therefore, if it was difficult for even the scientific world to assimilate the concepts of dimensions and spaces with higher dimensions, we can imagine that it took society in general far longer and a much greater effort to understand them, and when it happened it happened rather intuitively. The geometric revolution of the 19th century, which as we will see in the next chapter, went beyond the simple formulation of spaces with higher dimensions, was the key for human society to enter the world of multidimensional spaces.

Physical space versus mathematical space

There is an idea that was touched on in the last two sections, but which we have not yet gone into detail on, and that is the difference between physical space and mathematical space.

For physicists and other scientists, the concept of space is intimately linked to the concept of reality, but for mathematicians that is not the case. The question "Does four-dimensional space exist?" has a different meaning depending on who is asking it. For physicists, this question could be translated as "Could a real four-dimensional space exist?". The answer is clearly in the negative if we limit the concept of "real" to the observable physical world.

Hence, when talking about the fourth dimension, physicists tend to think in four-dimensional space-time. However, for a mathematician, the question could be rewritten as: "Could the concept of a four-dimensional space exist?"

Ultimately, the above distinction has a lot to do with the essence of mathematics and the way in which its exponents work. They, at least the pure mathematicians, observe the physical world that surrounds us, but are able to abstract themselves from it and travel to the world of ideas, concepts and mathematical structures, of which the physical world is just a small region, or entirely absent. Mathematicians work in that universe of ideas, obtaining abstract results, generalising concepts, creating new forms and tools. Despite the infinite distance between reality and mathematics, the latter finally ends up being applied, highly satisfactorily, in the real world. It is what Hungarian mathematician and physicist Eugene Wigner (1902-1995), winner of a Nobel Prize for Physics, called "the unreasonable efficiency of mathematics applied to natural sciences". In their famous book *Mathematics and*

Imagination (1989), mathematicians Edward Kasner and James Newman used another metaphor: "The mathematician is the tailor to the gentry of science. He makes the suits that anyone who fits into them can wear."

In this sense, mathematicians have worked naturally, and continue to do so, in multidimensional spaces, without bothering with the strictures of physical reality. For them, mathematical concepts exist as long as they are not logically conflicting. That is why, when someone talks to us about a four-dimensional space, we need not necessarily think of space-time or even of a fourth spatial dimension.

THE DIMENSIONS OF THE UNIVERSE

Our senses tell us that the universe we inhabit is a three-dimensional space, so, if we add time, we could consider that the universe is four dimensional. Nowadays physicists are developing string theory which suggests that our universe could exist in a space with many more dimensions, 10, 11 or even 26 of them, but that they are subatomic in size and thus beyod our ability to perceive them – or for many of us even imagine them! Interestingly, Charles Hilton already speculated about the chance that something like this could be possible in his discourse on the fourth dimension in the late 19th century.

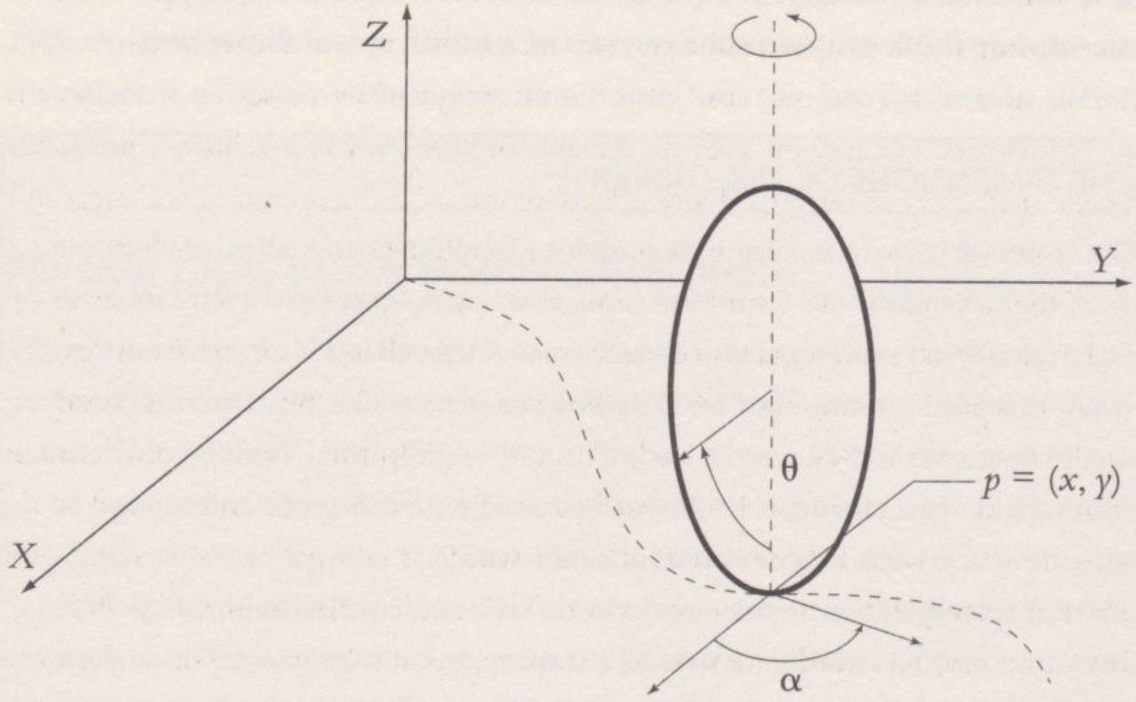
Although it has already caused a profound scientific and philosophical revolution, string theory has still not been proven experimentally. Its opponents say that it cannot ever be fully tested and therefore is not really a scientific theory at all. It is one of the things that the LHC (Large Hadron Collider) experiment at CERN may throw light on.

Are multidimensional spaces of any use?

In the world of mathematical physics the importance of working with spaces of higher dimensions became obvious very early on. In his book *Analytical Mechanics,* French mathematician Joseph-Louis de Lagrange (1736-1813) discussed mechanics in terms of many coordinates (degrees of freedom), including time as another coordinate. Subsequently, when Irish mathematician and astronomer William R. Hamilton (1805-1865) rewrote the mechanics equations, he did it within multidimensional spaces, so-called configuration spaces.

Let's look at an example. We need four coordinates to describe the space of the positions of a wheel that is moving forwards, without sliding, along a surface:

two to express the point of contact of the wheel on the surface, one for its angle of rotation and another for its angle of spin. That makes the space containing all possible configurations of the wheel a four-dimensional one. If we added movement – its dynamic – we would have to add another four coordinates for the velocities of the previous variables. Therefore, the space described by the dynamic of a wheel moving along a surface is eight dimensional.



This diagram illustrates how the space containing a wheel that is rolling – not sliding – on a flat surface has a fourth dimension. The coordinates are x, y, α, θ. The first two, x and y, give us the wheel's point of contact with the plane. α is the wheel's angle of spin, and θ its angle of rotation.

Most branches of science (physics, astronomy, economics, biology, medicine, engineering, etc.) make use of this kind of multidimensional space. The significance of working in these ways lies in the fact that it allows us to use geometric and mathematical tools to obtain useful information on the object under study or reveal interesting applications of it. Now follow two hopefully clear examples that show the usefulness of these techniques in our everyday lives.

Encryption of messages

Mobile telephones, Internet connections, digital televisions, music CDs, films on DVD, digital identification. All of these depend on the encryption of data and the subsequent deciphering of it, in which the detection and correction of errors is a crucial process.

In the digital age that we find ourselves in, to encrypt a message, be it an image, music or some text, requires translating the information into a series of zeroes and ones. This is called binary encryption (each 0 or 1 is known as a bit, a contraction of *binary digit*). Such a series is divided into 'words' of a fixed length, we will call it *k*. The 4-bit strings (containing 4 digits) are called hexadecimals. There are $2^4 = 16$ of them, while those with 8 bits are called bytes (there are $2^8 = 256$ versions). ASCII code which represents typed characters as bytes has 256 possible entries, that is to say that it can represent 256 different characters. The bits of each 'word' are normally treated as if they were coordinates, although in this case they only contain the numerical values 0 and 1. Each word formed by *k* bits represents a point in the coordinate space with *k* dimensions, in other words, it is equal to the length of the words. For example, the hexidecimal word 0011 is identified with point (0,0,1,1) of four-dimensional coordinate space. That space can also be given a distance, a way of measuring how far apart the points (binary 'words') of this particular geometric space are. For example, the so-called Hamming distance between two words measures the number of different digits in each (so the words **0**011 and **1**011 are at a distance of 1). Within this coordinate space we can make use of all mathematical tools: arithmetic, algebra, analysis, geometry.

However, things are not that simple, given that in the transmission of data – such as sending steering controls to a satellite or an email – or when reading the encrypted data (for example on a music CD) errors can occur. In this situation we have two problems: that perhaps we do not know if the information received is wrong and which bits are incorrect. Therefore, what we do is introduce error detector and corrector codes, increasing the length of the words and, consequently, the dimensions of the coordinate space.

An example of a code that detects errors is the Spanish tax identification numbers, which has a letter added to the identification number generated through a mathematical formula, in such a way that if we write it with one incorrect number, the letter will be different to what it should be, highlighting the error.

American engineer R.W. Hamming's error corrector code is a simple one, it adds another three bits to each hexadecimal word by means of a given mathematical equation (for example, the word 0011 becomes 0011101), and it is capable of correcting an error in one of the bits of the word being sent.

The Hamming code is very simple, but at the other end of the scale there are other error detection and correction codes, that are much more complicated, such as the Reed-Soloman code, which is used for CDs and in the telemetry of civil satellites, which use 65 and 265-bit words respectively. In other words, each word is a point in a coordinate space with 65 and 265 dimensions. As such, the use of mathematical tools in coordinate spaces is very useful, particularly in the creation of error detection and correction codes.

The Google™ search engine

Today, the Google™ search engine has become a fundamental Internet tool and has an mind-bogglingly large number of users. One of the reasons for such success is its efficiency, since when we ask it to search for a term, Google shows us an ordered list of resulting pages very quickly, and the first few will normally provide what we are looking for. The way in which Google orders (meaning to assign a numerical importance to each web page) the results of a search is a product of the use of mathematics, a mixture of linear algebra, graph theory and probability.

To design a search engine such as Google means tackling a mathematical engineering problem. Simply put, the main issue is how should we order our search results? The first model could be based on the assumption that the importance of a particular web page is related to the number of other pages that link to it. However, we could have a page which, although lacking in links, the ones it has are very important. This method would also be easily exploited by web sites wishing to dominate search results for their own benefits rather than those of the users.

What the creators of Google, Sergey Brin and Larry Page, did was to define the importance of a page, not in proportion to the number of pages that link to it, but proportionally to the importance of those pages. It translates into a system of algebraic equations that need resolving. In fact, for anyone who knows a little linear algebra, our problem has now become the calculation of eigenvectors and eigenvalues of a particular matrix. Underlying this is that the designation of the importance of web pages on the Internet $(x_1,...,x_n)$, where n is the number of pages existing on the network and x_i the numeric value that gives us the importance of web page i, translates into searching in a space with a number of dimensions equal to the number of web pages existing on the network for the item $(x_1,\ldots,x_n)$ which is the solution to the previous series of equations. In 2006 it was estimated that there were around 600 billion web pages, a figure that corresponds to the dimension being considered. That certainly is a multidimensional space!

THE ALGORITHM THAT CHANGED THE INTERNET

In 1998, two young computer students at Stanford University in California, Larry Page and Sergey Brin, were putting the finishing touches on a research project with the somewhat cryptic name of *The Anatomy of a Large-Scale Hypertextual Web Search Engine*. In it was contained the first draft of *PageRank*, a simple and elegant algorithm used for prioritising any list of pages based on their relevance. *PageRank* became the backbone of the Google search engine, which a few years later would overtake Yahoo, Altavista and many others in the preferences of the billions of Internet users. Googling even became a synonym for searching the web

PageRank is extremely elegant and simple, and its operation can be formulated as follows:

$$W_j=(1-d)+d\sum_{i=1,i\neq j}^{N}\frac{W_i}{n_i}$$

where W_j is the value of page *j*; W_i, that of page *i* that includes a link to page *j*; *d*, a damping factor, of a value between 0 and 1, used to ensure the convergence of the series; n_i, the number of links on page W_i to other pages, and *N*, the total number of pages that include a link to page *j*.

The relevance of any page is the result of the sum of the relevance of all the pages, which link to it, weighted by the total number of links on each.

Chapter 3

The geometric revolution of the 19th century

> *The fundamental hypotheses of geometry are not experimental facts. It is, however, the observation of certain physical phenomena that accounts for the choice of certain hypotheses among all possible ones. On the other hand, the chosen group is only more convenient than the other possible ones, and we can no more say that Euclidean geometry is true and Lobachevski's geometry is false, than we could say that Cartesian coordinates are true and polar coordinates false.*
>
> *H. Poincaré,* On the Fundamental Hypotheses of Geometry *(1887)*

It is not often that mathematical issues become of general interest. However, the fourth dimension, after the double geometric revolution that took place in the 19th century, penetrated deeply into society. It peaked the interest of scientists and philosophers, theologists and mediums, writers and artists, musicians and poets and the public at large.

Non-Euclidean geometries

In around 300 B.C., Euclid of Alexandria published his great work *Elements of Geometry*, a compendium of all geometric, arithmetic and algebraic knowledge at the time. It was structured, starting with the initial elemental issues, as an ordered display of knowledge that used the deductive method and the value of demonstrating proofs, but in which more informal reasoning, such as intuition, analogy and symmetry, among others, also played an important role.

After religious books, *Elements* has possibly been the most influential book of all time. It was repeatedly hand copied, translated into many languages and, after the invention of printing, constantly republished. For more than two millennia it has been used as a text book and was a model for mathematical thinking.

One of the merits of Euclid was knowing how to select a group of fundemental postulates from which, by means of the axiomatic-deductive method, all knowledge could be constructed. Thus, when the study of plane geometry began, it presented some definitions of point, straight line, angles etc that are more descriptions of intuition; axioms that are truths considered to be self evident and not needing further demonstration. For example, 'two things equal to the same thing are equal to one another' or 'the whole is greater than the part'. Euclid's five postulates form the basis of his system, although he does not demonstrate or prove them:

1. A straight line segment can be drawn joining any two points.
2. Any straight line segment can be extended indefinitely to form a straight line.
3. Given any straight line segment, a circle can be drawn having the segment as radius and one end point as its centre.
4. All right angles are congruent.
5. If two lines are drawn that intersect a third in such a way that the sum of the inner angles on one side is less than two right angles, then the two lines inevitably must intersect each other on that side if extended far enough.

EUCLID OF ALEXANDRIA

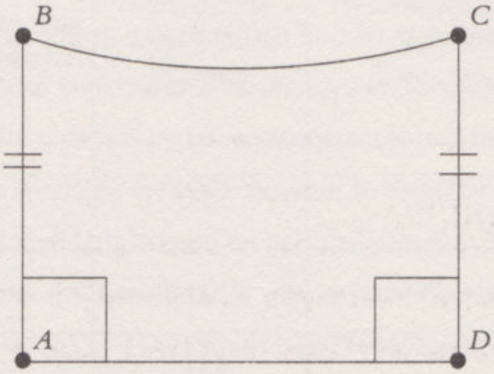
It is perhaps surprising that so little is known about the life of the author of *Elements*. He was known as Euclid of Alexandria – there were others from elsewhere – because he was a professor at the Museum in the city, an institution that, together with the magnificent Library, sought to be a repository of all knowledge.

Euclid was modest and friendly although often sarcastic. "There is no royal road to geometry" was his response to Ptolemy, the city's ruler, who was after a short cut to understanding geometry, without having to actually read the master's book. And when a student asked why he should study geometry, he told a slave to give him three coins "since he needs to gain from what he learns". Euclid is thought to have written also on a wide range of other subjects, such as optics, astronomy, geometry, music and didactics, although historians are unsure whether the same Euclid is the author of all the texts attributed to him.

It is evident that the fifth postulate, whose modern form is "Through a point not on a given line, only one line can be drawn that never meets the given line", is not written with reference to the previous ones. It is also longer and made in the form of a proposition. Some mathematicians thought that it could be deduced from the previous axioms and postulates and tried to demonstrate it. Many of them died believing that they had achieved it, and others questioned whether it was necessary to consider it a postulate at all. For more than two millennia many illustrious mathematicians became involved in the resolution of the fifth postulate or parallels problem.

The point of inflection in solving this question came from the Italian mathematician Girolamo Saccheri (1667-1733). Instead of trying to solve the problem by directly demonstrating the fifth postulate by means of the previous ones, it was proposed that it be proved to be true using the *reductio ad absurdum* method. The argument is based on this birectangular quadrilateral. For angles B and C, which are equal, there are three possibilities:

a) = 90° (right-angle or Euclidean hypothesis);
b) > 90° (obtuse angle);
c) < 90° (acute angle).



Saccheri's birectangular quadrilateral.

The obtuse angle hypothesis is quickly discarded, but he only discards the acute angle hypothesis when there is a result that he considers inappropriate, although not contradictory: "The hypothesis of the acute angle is absolutely false, because it is repugnant to the nature of the straight line." Both Saccheri and German mathematician Johann Heinrich Lambert (1728-1777) obtained some interesting geometric results under the acute angle hypothesis.

But it was not until the 19th century that Gauss, Lobachevski and Bolyai resolved the problem definitively, although German mathematician Johann Carl Friedrich Gauss never published his discoveries as they contradicted the era's restrictive philisophical doctrines on the nature of space.

It was Russian mathematician Nikolai I. Lobachevski, who was first to divulge an extensive geometry that was different to Euclid's, he called it "imaginary geometry" and it is now known as hyperbolic geometry. This corresponds with Saccheri's acute angle hypothesis, according to which infinite parallel lines pass through every external point of a given straight line. Lobachevski made his work public in 1826, at a conference held at his university, and subsequently in *The Kazan Messenger* magazine, in an article called *About the Principles of Geometry*, which would be followed by others. Three extensive works bring together his new geometry, the titles of which in English would be *New Fundamentals of Geometry* (in Russian), *Geometric Investigations on Parallels* (in German) and his last book, *Pangeometry* (in Russian and French).

For his part, Austro-Hungarian army officer and mathematics enthusiast János Bolyai (1802-1860), approached the problem from a slightly different angle: he

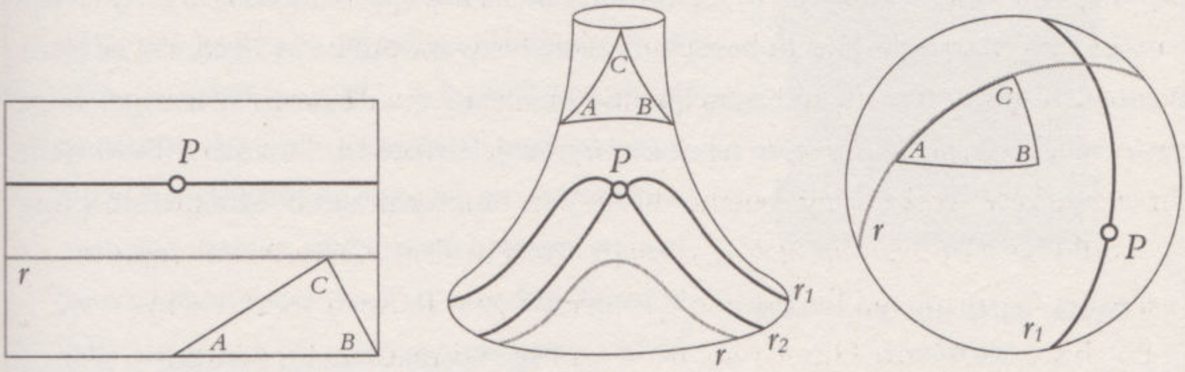
IMMANUEL KANT AND EUCLIDEAN GEOMETRY

After the Renaissance, the figure of God begun to lose its importance in mathematics and science in general. His image as mathematician and architect of the world faded further in the 18th century. It is said that Napoleon reproached French mathematician Pierre Laplace (1749-1827) that in the volumes of his work *Celestial Mechanics* he did not mention the Creator, to which Laplace replied: "Sir, I had no need of that hypothesis".

But then philosophers asked themselves why the mathematical laws of nature had to be true in the first place. Scottish philosopher David Hume (1711-1776) believed that our knowledge of the world is subjective, since it comes through our senses. In other words nobody could guarantee the existence of an objective physical world and, consequently, it did not make sense to talk about its scientific laws.

For his part, Kant, in his work *Critique of Pure Reason* (1781), states that space and time are types of perception, intuitions based on which the mind considers the experience. Given that the intuition of space is in our minds, it accepts certain truths about space, which Kant calls "*a priori* synthetic truths" that form part of our innate mental faculties. Geometry simply explores its consequences. Euclidean geometry and three-dimensional space form part of those *a priori* – before the fact – truths.

developed an absolute geometric theory by working only with the first four postulates and studying whether or not the geometric results obtained depended on the fifth. This work was published in 1832 as an appendix to a work written by his father, mathematician Farkas Bolyai (1775-1856), a close friend of Gauss, who had also worked on the parallel problem. He wrote this to his son: "For God's sake, please give it up. Fear it no less than the sensual passion, because it too may take up all your time and deprive you of your health, peace of mind and happiness in life."



Euclidean geometry $A + B + C = 180°$.

Hyperbolic geometry $A + B + C < 180°$.

Elliptic geometry $A + B + C > 180°$.

Geometry	Lines parallel to a straight line pass through external point *P*	Sum of the angles of a triangle	Circumference of a circle with diameter 1	The square of the hypotenuse of a right-angled triangle with sides a and b
Euclidean or parabolic	One	180°	$= \pi$	$= a^2 + b^2$
Hyperbolic	Infinite	< 180°	$> \pi$	$> a^2 + b^2$
Elliptic	None	> 180°	$< \pi$	$< a^2 + b^2$

Both the sum of the angles of a geodesic triangle and the number of lines parallel to a straight line that pass through an exterior point depend on the type of geometry, either Euclidean, hyperbolic or elliptic.

Initially, almost nobody was interested in the somewhat marginal work of this pair. Lobachevski's work was largely written in Russian, and Bolyai's was published as an appendix. The mathematical community only began to show interest after German mathematician Bernhard Riemann's dissertation *On the Hypotheses that Underlie Geometry* (1854), which is discussed later. Riemann was the first mathema-

NIKOLAI IVANOVICH LOBACHEVSKI (1792-1856)

ПОЧТА СССР 1951г.
40 КОП.
Н.И.ЛОБАЧЕВСКИЙ
ВЕЛИКИЙ РУССКИЙ ГЕОМЕТР

Russian stamp with a portrait of Lobachevski.

The father of non-Euclidean geometry was a man of humble beginnings; an extremely well-mannered and serious, a tireless worker who was committed to university life and lived nearly all his life in the Russian city of Kazan. After graduating in physics and mathematics from the city's university, he began to work there and quickly reached the position of dean of his faculty and, finally, that of Rector of Kazan University, a post he filled for 19 years. While still working on mathematics, he did some exceptional work in this role. He improved the buildings and built new ones, efficiently managed the library (occasionally he could be seen personally organising the books), had laboratories and a new clinic built and conditioned, attracted better teaching staff and academics. As well as his work in geometry, Lobachevski was also interested in other fields of mathematics, such as trigonometric series, probability, mechanics and integral calculus, although his most important non-geometric work was *Algebra or Calculus of Finites.*

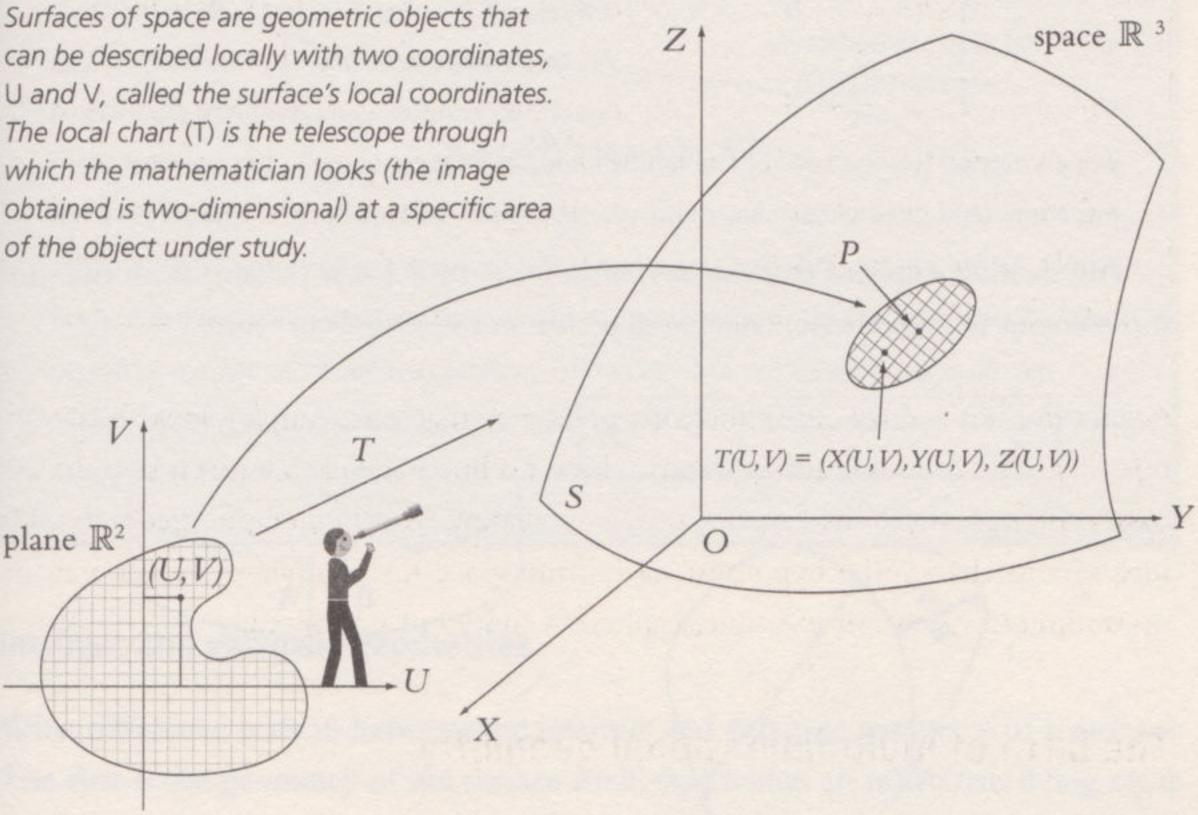
tician who noticed the possibility of a geometry that was compatible with the obtuse angle hypothesis – according to which no line parallel to a given straight line passes through any of its exterior points – creating so-called elliptic geometry. His idea was to change the hypothesis of infinite space for another more general one on unlimited space. For example, a sphere is finite but unlimited.

The birth of multidimensional geometry

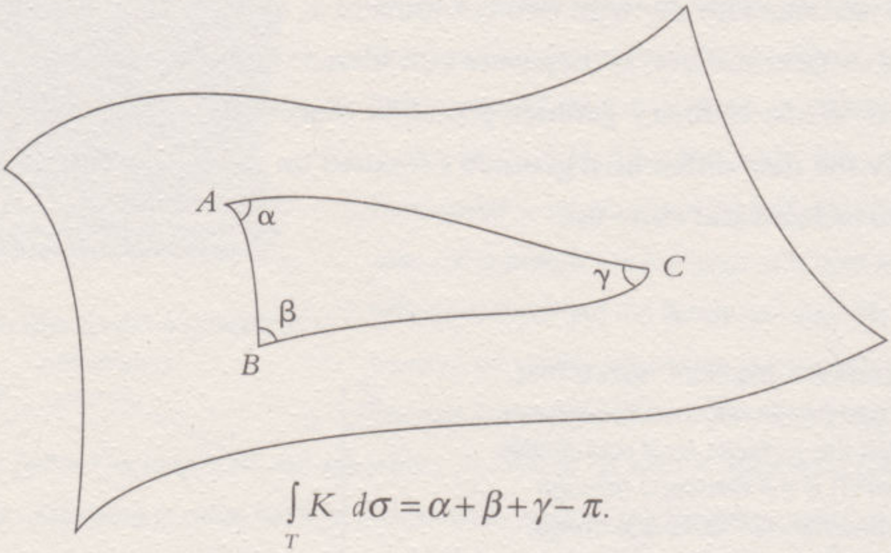
In 1827, with the publication of Gauss's *General Investigations on Curved Surfaces,* a new branch of geometry was born, differential geometry, that consists in the use of differential and integral calculus in the study of curves and surfaces in three-dimensional Euclidean space. Since the origin of calculus, from the work of Newton and Leibniz, mathematicians had made use of this powerful tool for analysing curves and, subsequently, Euler and Monge began to use it for surfaces, too.

However, even Gauss's work did not contain a systematic and exhaustive study of the surfaces of three-dimensional space. Gauss became interested in the study of surfaces when he came across problems in his study of geodesy and cartography, the origins of which can be traced back to his triangulation measurements in Hanover, as well as other issues that came up in his astronomy research. In *Investigations,* while studying surfaces of geometric spaces themselves, he established a new method of study. He was the first to reveal the convenience of thinking of the surfaces as objects that can be described locally with two coordinates x_1 and x_2, called the local coordinates of the surface, and not, as they were considered until then, a mere boundary of solid objects. While ordinary geometry studied objects on a plane and in space in their totality, the new differential geometry focused on the local properties of curves and surfaces isolated and close up.

Surfaces of space are geometric objects that can be described locally with two coordinates, U *and* V, *called the surface's local coordinates. The local chart* (T) *is the telescope through which the mathematician looks (the image obtained is two-dimensional) at a specific area of the object under study.*

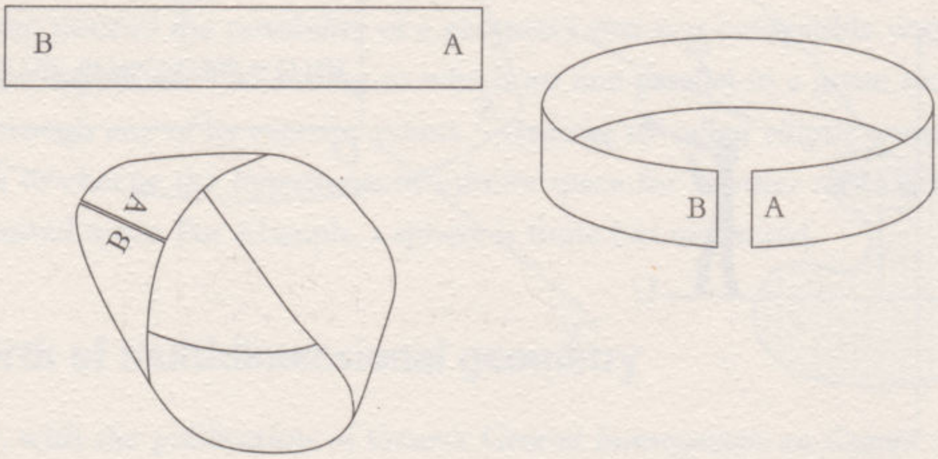


In the above-mentioned work, Gauss introduced the concept of the orientation of a surface and the associated field of normal vectors, which is a vector perpendicular to the surface at all of its points, and which becomes a fundamental tool for measuring the curvature of the surface by simply observing how it varies around each point. Using this tool, two curvatures are developed for the surface, known today as the Gauss curvature K, and the measured curvature H. Gauss showed that, contrary to what was seen in the definition, the first is an element of the surface's intrinsic geometry (the so-called *Theorema Egregium*). He also introduced other fundamental elements of intrinsic geometry, such as geodesics (which are lines that make up the shortest distance between two points on a curved surface), and obtained interesting results from this geometry, such as the relationship between the angles of a geodesic triangle and its curvature:



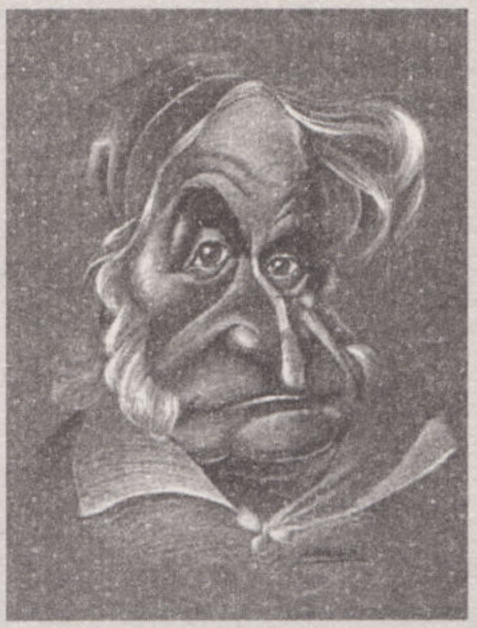
$$\int_T K \; d\sigma = \alpha + \beta + \gamma - \pi.$$

The formula explains that the deviation from 180° (or, π radians) from the sum of the angles of a (geodesic) triangle depends on the Gauss curvature.



If we take a strip of paper and connect its two ends we get a surface with two faces, the exterior face and the interior face; therefore it is double sided. But if we turn one end of the paper over, we get a Möbius strip, which only has one face. To check this we can draw a line along the strip with a pencil and when we return to the start we will have covered the whole strip, which is a single-sided object.

JOHANN CARL FRIEDRICH GAUSS (1777-1855)

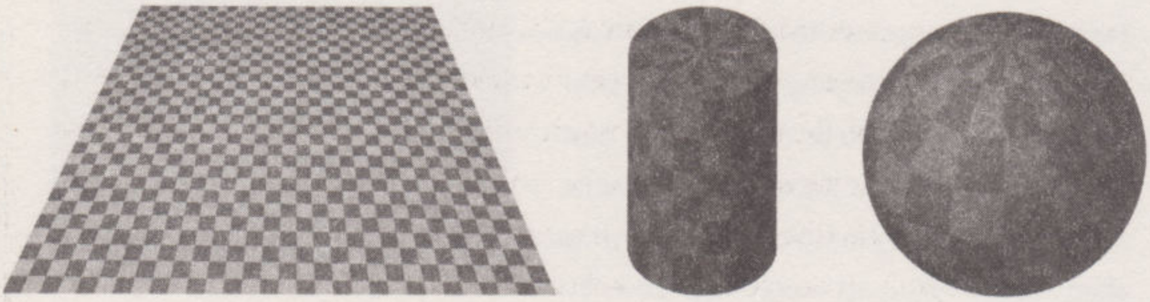

Caricature of Gauss by Enrique Morente.

Gauss is undoubtedly one of the most outstanding mathematicians of all time. When he was a child he showed an exceptional talent for mathematics, which meant his education was funded by Duke Wilhelm Ferdinand, despite his humble origins. Thus, in 1795 Gauss began to study mathematics at the University of Göttingen. At the age of 19 he solved one of the classic problems of geometry by showing that a regular polygon with 17 sides could be drawn using a ruler and compass. This was the first entry in his famous scientific diary in which he would write short notes on his most important discoveries. Between 19 and 21 years of age he wrote his masterpiece *Disquisitiones Arithmeticae*. He became famous across Europe when he determined the orbit of the asteroid Ceres on paper, using his method of least squares. In 1807 he was appointed head of astronomy at the University of Göttingen and was named director of the observatory. He made innovations in many fields of mathematics, including algebra, number theory, differential geometry, non-Euclidean geometry, mathematical analysis, geodesy, astronomy, error theory, as well as in physics, magnetism, optics and electricity. The title of the Prince of Mathematics was given to him by King George V of Hanover on coins minted in honour of Gauss after his death.

Intrinsic and extrinsic geometries

What difference is there between the intrinsic and extrinsic geometry of a surface? The first is the geometry of the surface itself, that which an inhabitant living on it could describe. Gauss mentioned hypothetical moths living in the two-dimensional space in his letters to his colleagues. Gauss's *Theorema Egregium* establishes that the Gauss curvature is part of the geometry that is intrinsic to the surface. It measures the way in which it curves internally. However, extrinsic geometry is concerned with the relationship between the surface and the exterior three-dimensional space, and the measurement of how the curves in it relate to the average curvature.

Locally, the intrinsic geometry of a plane and a cylinder is the same, given that both have a Gauss curvature of zero. If we take a sheet of paper and join the two opposite ends we get a cylinder, in this small experiment we have changed the geometry (the distances) of the surface. Both surfaces are intrinsically flat, and an inhabitant of them would not be able to tell one from the other, unless they could look from or towards the outside, given that the plane is not curved in three-dimensional space (its average curvature is zero), but the cylinder, the average curvature of which is constant and positive, is curved.



Plane ($K = 0$, $H = 0$); *cylinder of radius* r ($K = 0$, $H = 1/r > 0$); *sphere of radius* r, ($K = H = 1/r^2 > 0$).

Also, the intrinsic geometry of the sphere, the Gauss curvature of which is constant and positive, is different from that of the plane, which is the reason why an inhabitant of the sphere could realise that they are living in a curved space without having to go outside it. They could do this by verifying the sum of the angles of a geodesic triangle is greater than 180°, as Gauss tried to prove with the Earth, although the deviation resulting from its measurement was within its own margin of error. Moreover, a significant consequence of this is the impossibility of making correct maps of the Earth, as they should conserve the geometry (distances, shortest paths, areas or directions). However, for most surfaces the value of Gauss's curvature varies from one point to another. An example is the surface of the torus, or ring doughnut, which has points with positive, negative and zero Gauss curvature (the exterior, interior and intermediate circumference points respectively).

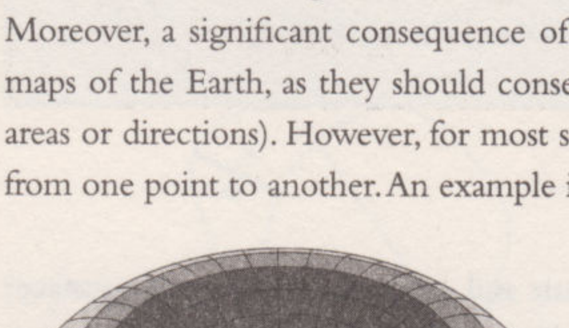
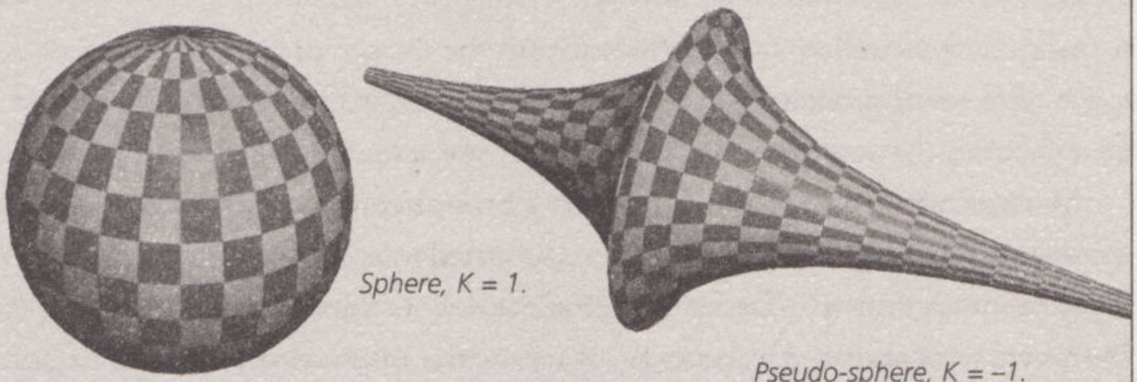


Image of the surface of a ring, called a torus, highlighted with different colours based on whether the curvature is positive, neutral or negative.

MODELS OF GEOMETRY ON SURFACES

In the search for models of non-Euclidean geometries it can be considered that spaces are the surfaces, and its geodesics (remember that these are lines that form the shortest distance between two points on a curved surface) are the 'straight lines'. Differential geometry tells us what the surfaces that verify Euclid's postulates are like. They must be geodesically complete (its geodesics are unlimited) so that postulates 1 and 2 are verified, and have a constant Gauss curvature K, so that postulates 3 and 4 are verified. So, if K = 0, we have Euclidean geometry, a plane. If K > 0, we have a hyperbolic or obtuse-angle geometry model, such as a sphere – in these cases the first postulate is not verified, since infinite geodesics pass through antipodal (opposite) points, but this can be solved by identifying the antipodal points of the sphere, although the result is an abstract surface, outside the three-dimensional Euclidean space. If K < 0, this is an elliptic or acute-angle geometry model, the pseudo-sphere; this model is not fully effective either, since it is not geodesically complete and, therefore, we again have to look for models outside of the three-dimensional Euclidean space.



Sphere, $K = 1$.

Pseudo-sphere, $K = -1$.

Riemann's contribution

In any case, the revolution begun by Gauss still kept within Euclidean three-dimensional space. The multidimensional adventure was yet to begin. Until then, ordinary analytical geometry was dedicated to the study of (coordinate) spaces of up to three dimensions (straight lines, planes and space) and, as we have already said, admitting the existence of higher dimensions was not easy for scientists or philosophers. However, around the middle of the 19th century, multidimensional spaces were introduced as a natural extension of analytical geometry. Two important works related to this were the article *Chapters on the Analytical Geometry of* n *Dimensions*, by

English mathematician Arthur Cayley (1821-1895), and *Lectures on Linear Extension*, by German mathematician and philosopher Hermann Grassmann (1809-1877).

Then came Riemann's essay which was to be presented at Göttingen University, *On the Hypotheses that Underlie Geometry*. It contained some grand geometric ideas:

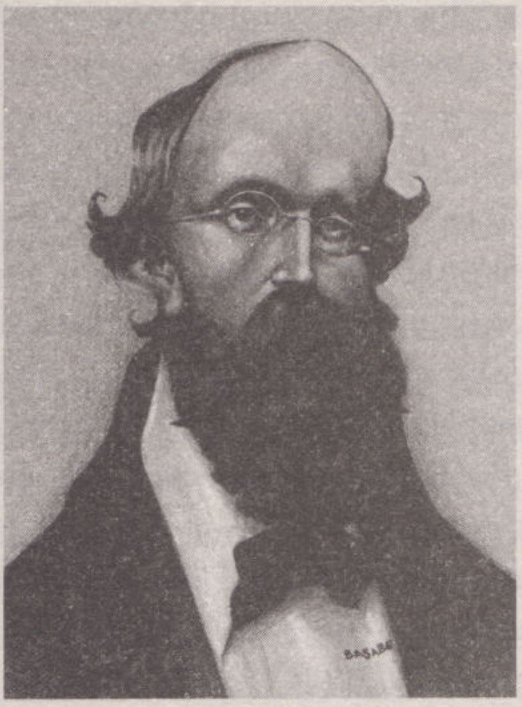
1. The introduction of n-dimensional geometric spaces (which geometricians call differentiable manifolds), which generalise the concept of surface introduced by Gauss.
2. The introduction of a metric tensor that generalises the concept of distance and the study of metric relationships on differentiable manifolds (the birth of Riemannian geometry).
3. The generalisation of curvature and other elements of intrinsic surface geometry to Riemann's n-dimensional manifolds.

The definition of the concept of n-dimensional differentiable manifolds includes the fact that we can identify it locally with n magnitudes, the local coordinates, $x_1,\ldots,x_n$, as well as the rules that govern the change of coordinates. Geometric space (differentiable manifold) is not necessarily linked to a spatial reality, but can be any entity that verifies the general conditions established by the definition.

Moreover, Riemann broke away from previous mathematical and philosophical thinking, according to which the concept of space was implicit to distance, understood as usual distance (Euclidean). With this, he separated the concept of space (n-dimensional differentiable manifold) and that of distance (termed Riemann's metric tensor), in such a way that in the same space we can find three distances, which, of course, would have different associated curvatures. Therefore, Riemann's geometry is a non-Euclidean geometry in a much more general sense than that which was developed by Lobachevski and Bolyai, as it introduces higher dimensions, and the curvature can take on different values at different points.

On the other hand, Riemann also stated a profound interest in physics problems, which led him to attempt to unify the physical forces of nature (such as gravity, electricity and magnetism). In his opinion, force was a consequence of the geometry of space and its curvature, and the new geometry he introduced could provide him with the unification of the natural forces he craved.

BERNHARD RIEMANN (1826-1866)



Caricature of Riemann by Gerardo Basabe.

Riemann had a short life and published few works, but those which he did publish were of exceptional quality, since in them he resolved some of the most intricate problems in mathematics. He also introduced new concepts and tools, and profoundly changed the idea of space for scientists and philosophers. He was a shy person terrified of public speaking. He also had delicate health and suffered frequent nervous breakdowns. He had a humble childhood, as might be expected from the child of a shepherd, but he had a fantastic ability for calculation and a special talent for mathematics. It is said that during his secondary school education he was given Legendre's *Theory of Numbers*, devouring the 900 pages in one week. Although he began his studies in theology and philosophy, Riemann was quickly attracted to mathematics, so he went to study it at the University of Berlin where he began to develop original ideas on functions of complex variables, the subject on which he submitted his doctoral thesis in Göttingen, under the tutorship of Gauss. In 1859, Riemann wrote his only publication on primary numbers, a subject that would captivate him for years, and in which his most famous hypothesis appears.

The ideas that he left behind were fundamental in the physics of the 20th century. In particular, they were the precursors to the theory of relativity: German physicist Albert Einstein (1879-1955) introduced the theory of restricted relativity in 1905, together with Dutch physicist and mathematician Hendrik Lorentz (1853-1928) and French mathematician Henri Poincaré (1854-1912). Shortly afterwards, German mathematician Hermann Minkowski (1864-1909) incorporated a four-dimensional Riemann variety, space-time, with a spatial Riemann metric tensor which involved the speed of light and this was the space in which Einstein's theory of general relativity, introduced in 1916 could best be developed.

From blackboard to coffee shop

The beauty of the ideas revealed in Riemann's dissertation soon spread to all of Europe's education and research establishments. Multidimensional differential geometry, as well as non-Euclidean geometries, were beginning to gain popularity in the mathematical and scientific world. The research continued. In terms of non-Euclidean geometries, work was done on their consistency, so that the accepted premises no longer contained logical contradictions, as well as obtaining new models of space. In differential geometry the edifice begun by Riemann continued to be built, decorated and furnished, at the hands of great mathematicians, such as the Italians, Eugenio Beltrami (1835-1900), Gregorio Ricci-Curbastro (1853-1925) and Tullio Levi-Civita (1873-1941), and also the German Elwin Bruno Christoffel (1829-1900). Scientists of the time began to try to apply this new and elegant theory, and although at first it was not easy (physics, for example, needed to be developed further first), the 20th century demonstrated the true stature of this relatively recent branch of geometry.

At the same time, mathematicians and scientists of the era began the sizeable task of spreading the word about non-Euclidean and Riemann's geometry within the academic world through conferences, articles in scientific magazines and books, etc., and little by little they began to introduce the ideas to the general public.

One of the most prolific evangelists of the fourth dimension was German mathematician Hermann von Helmholtz (1821-1894). His articles were published in

German physicist Hermann von Helmholtz wrote and spoke extensively on non-Euclidean geometry and hypothetical multidimensional worlds. His proposals struck a cord with the public and became well-known throughout the world.

Germany, France, Britain and the USA in the 1860s and 1870s. Helmholtz, like a few of his contemporaries would also do, alluded to sentient bi-dimensional beings that lived on a sphere and also other surfaces. These creatures would develop their own geometry, different to Euclidean geometry, in which, for example, the sum of the internal angles of a triangle would not be 180°. However, regarding the fourth dimension, Helmholtz considered in his work *Popular Lectures on Scientific Subjects* (1881) that it is impossible for us to visualise it, and makes a comparison with the fact that a person who is born blind cannot imagine colours.

At the same time as some scientists were working on profound questions, others were busy with other more mundane problems – How do two-dimensional beings eat? What is their digestive tract like? How do they get around? What would their eye be like, and consequently, their vision? – and other similar questions which, of course, attracted a more public attention. In those times the expression *fourth dimension* became a synonym for any multidimensional space, and on occasion non-Euclidean and multidimensional geometry were confused.

The profundity of these geometric revolutions led to their inclusion in some of the most important scientific and philosophical debates at the end of the 19th and beginning of the 20th centuries. Noteworthy among them are questions on truth in science, the relationship between existence, science and reality, the possible existence of higher dimensions, and the structure, operation and importance of mathematics. The concept of space also came under scrutiny and, above all, a new question was posed: Is our space Euclidean or non-Euclidean?, or more commonly, what shape is our space?

The popularisation of the fourth dimension also brought with it surprising, even magical, aspects, as we will see in Chapter 4. It suggested hyperbeings that were omnipotent and omnipresent, capable of passing through walls and all kinds of impressive superpowers. This inevitably led to higher dimensions becoming subjects concerning religion and spiritualism, and more generally, faith. Four-dimensional space could be seen as evidence of gods or the supernatural. For example, Christian thinkers suggested that God and immortality could be connected to our three-dimensional visible world by a fourth dimension.

But the great propagation of the subject of the fourth dimension to society in general came in 1877, with a scandalous court case held in London, which was reported in both the British and international press. Henry Slade, a recently arrived American medium, was standing trial for fraud, after having carried out spiritual sessions with important people from London society. But the scandal heightened when

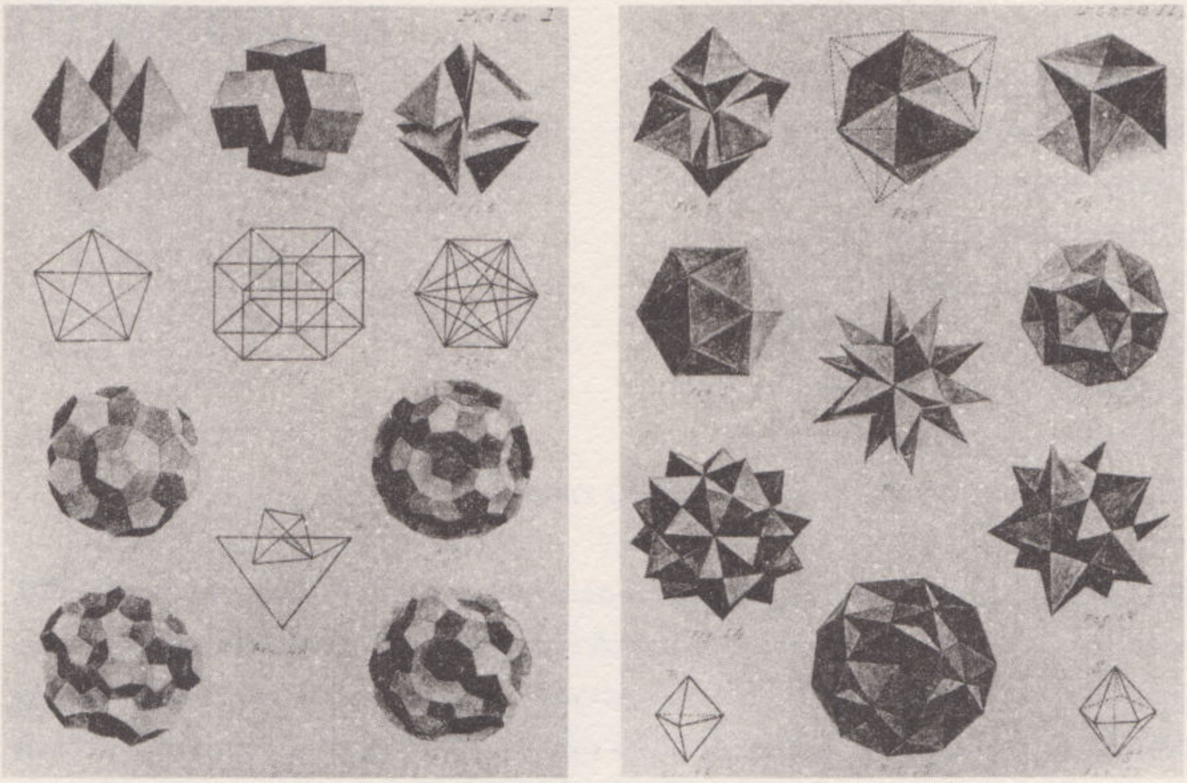
Henry Slade was one of the most popular mediums of the 19th century, and when his sessions of spiritualism were described as fraudulent, part of the scientific establishment came to his defence.

eminent scientists from the world over (including a few who would go on to win Nobel prizes) came out in his defence, assuring that his sessions demonstrated that spirits were really beings of the fourth dimension. Despite the accusation of Slade, Johann K.F. Zöllner (1834-1882), professor of physics and astronomy at Leipzig University, carried out a series of experiments in order to demonstrate this as we will see in Chapter 5. This scandal made multidimensional spaces, although a completely non-scientific version of it, a topic of conversation across Britain and the wider world.

Another fourth-dimension related issue that became very popular was the game of trying to visualise objects in it. One of the first scientific works to tackle this problem was the study of hypersolids carried out by American mathematician Washington Irving Stringham (1847-1909) in his article *Regular Figures in N-Dimensional Space* (1880), which had a great impact. Particularly, attempts to visualise the hypercube, the fourth-dimensional generalisation of a three-dimensional cube, became a symbol of the visualisation of the fourth dimension, and Charles H. Hinton was one of the people that dedicated great efforts to it, being convinced that it was possible to visualise the fourth dimension, as was Poincaré. Hinton was the principle representative of what became known as the 'philosophy of hyperspace', a popular philosophy that reflected on higher-dimensional spaces and their interaction with other materials.

The fourth dimension also became a subject used by some of the era's writers. Philosophically speaking, after the discontent with materialism and positivism, multidimensional spaces and non-Euclidean geometries contributed to the development of several cultural phenomena.

In the world of art, it allowed cubists to break away from the Renaissance perspective and paint the object from different points of view. Likewise, musicians, designers, architects and artists in general were talking of a new language of art and approaching higher realities. The fourth dimension was penetrating all social and cultural arenas, and it was not unusual for it to be a topic of conversation in coffee shops, mixing mathematics with the usual diet of gossip and politics.



Some of the Regular Figures in *n*-Dimensional Space*, by W.I. Stringham, published in the American Journal of Mathematics in 1880. The images correspond to figures 1 and 2. The top three images of figure 1 represent the "faces" of each vertex which we can call hypertetrahedron, hypercube and hypericosahedron, which extend the tetrahedron, the cube and the icosahedron into the fourth dimension. Thus, in the case of the hypertetrahedron, each vertex touches four tetrahedrons, in the same way that for a three-dimensional tetrahedron each vertex touches three triangles. For the hypercube, each vertex touches four cubes, in the same way that for a cube each vertex touches three squares. The next row represents the projections of the previous three four-dimensional figures onto a plane.*

Chapter 4

The magic of the fourth dimension

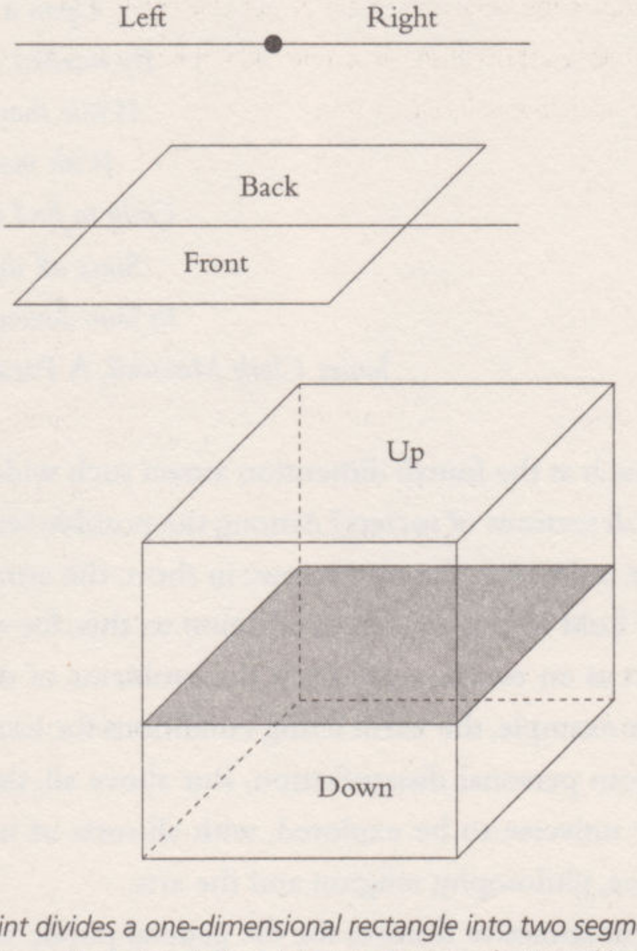
My soul's an amphichiral knot,
Upon a liquid vortex wrought
By intellect in the unseen residing,
While thou dost like a convict sit
With marlinspike untwisting it
Only to find my knottiness abiding,
Since all the tools for my untying
In four-dimensioned space are lying.
James Clerk Maxwell, A Paradoxical Ode (1878)

Why did an issue such as the fourth dimension attract such wide attention from not only scientists but all sections of society? Among the possible reasons, we could suggest the lure of the unknown, the mysterious; in short, the attraction of something that we can never hold in our minds. In addition to this, for some people, higher dimensions can act as an escape valve from the problems of the society in which they live (recall, for example, the harsh living conditions for most in Victorian England), or simply, from personal dissatisfaction. But above all, the fourth dimension represented a new universe to be explored, with all sorts of implications for new directions in science, philosophy, religion and the arts.

Among the most attractive features for the general public were those related to faith, especially for those with interests in spiritualism. These are covered in Chapter 5. There were, however, other surprising and perhaps even magical or miraculous, aspects of the fourth dimension that aroused the public imagination and we shall take a closer a look at these in this chapter.

Seeing from the fourth dimension

Imagine that our three-dimensional universe formed part of a four-dimensional hyperspace. Under such circumstances, such hyperspace would be divided in two, using what Charles Hinton calls "ana" and "kata". Just as a point in dimension zero divides the straight line into two semi-lines, 'right' and 'left', a straight line divides a plane into two semi-planes, 'front' and 'back'. The plane divides space into two semi-spaces, which we can refer to as 'up' and 'down', although, as in other cases, which is which is simply a matter of choice. In general terms, an *n*-dimensional hyperspace will divide an $n + 1$ dimensional hyperspace into two semi-hyperspaces.

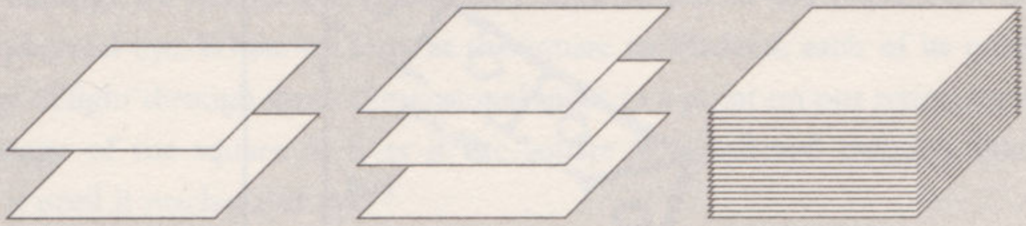


A zero-dimensional point divides a one-dimensional rectangle into two segments, left and right. A straight line divides a two-dimensional plane into two regions, front and back. A plane divides a three-dimensional space into two semi-spaces, up and down. Similarly, a three-dimensional space will divide four-dimensional hyperspace into two distinct regions, ana and kata.

Some Christians with an interest in the fourth dimension have seen this system as a way to explain the location of heaven and hell. Heaven, with God and his cohorts of angels, is located on one side of our visible universe, for example in ana, whereas hell, home to the Devil and his assembled demons, is located in kata. In other words, angels and demons are separated by our earthly world.

PARALLEL UNIVERSES

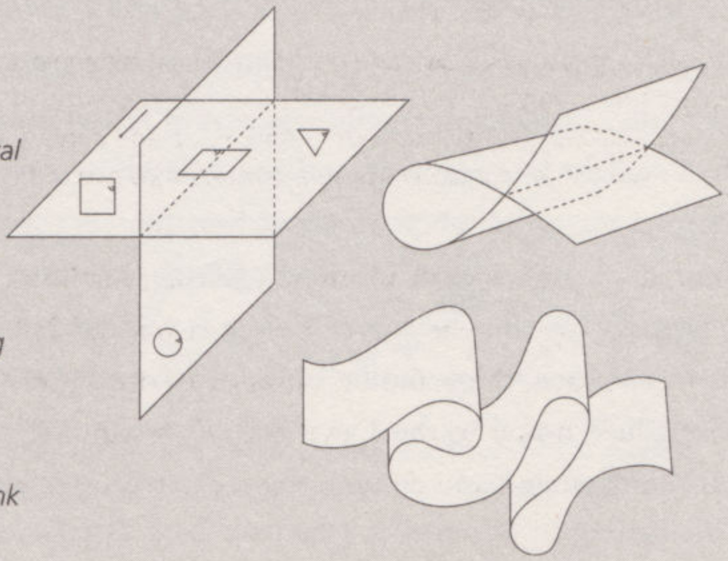
We can speculate on the possibility that our universe is not alone in hyperspace and that there are other parallel universes, from the simplest situation of two parallel universes, the physical and the astral, passing through the Christian version of three parallel universes with heaven, hell and the Earthly world, all the way to infinite parallel universes, which hold all possible variations of our own. In one of these, a mathematician is writing a book on the fourth dimension, but in another, the same individual opts for philosophy and German classes, or in a third, he simply does not exist because his parents did not have children. It is even possible to have a universe in which people fly with wings.



Two universes Three universes Infinite universes

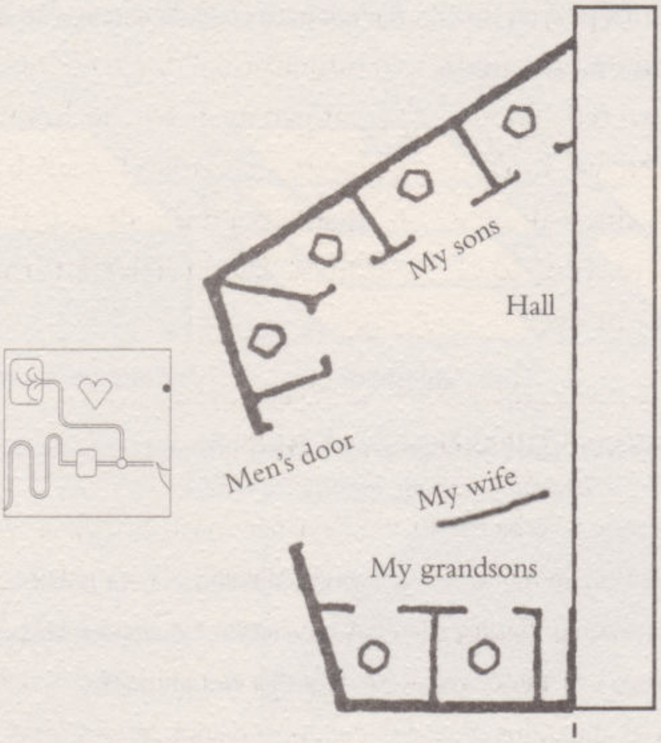
Parallel universes: Two universes (ours and the astral), three universes (our world, together with heaven and hell), infinite universes.

There could also be perpendicular universes, where the intersection – or portal – of two would be a plane. Another possibility is that our universe is curved in hyperspace and even intersects itself, thus creating portals – or wormholes – between far-distant points.



Two universes in four dimensional space can be perpendicular as well as parallel, thus creating a portal between them. Moreover, a universe does not have to be 'flat', but it can also be curved in hyperspace and even intersect itself, creating portals through which it would be possible to travel between different parts of the same universe in the blink of an eye.

Under such circumstances, we can ask how a being from the fourth dimension such as an angel, which would live in ana, could appear to beings such as us who lived in three-dimensional space. To answer this question, we will harness the three-dimensional analogy set out by Abbott in *Flatland*, and we will ask how the sphere appeared to the square during its visit. As the illustration below shows, the sphere, observing from Spaceland, sees the full perimeter of the square and all its interior in the same view, just as its eyes see the outside of a Flatland square's home and also the family members that live there.



When observing Flatland from Spaceland, we can see the square enter his house and all the beings who are there. We can also see both inside and outside the square.

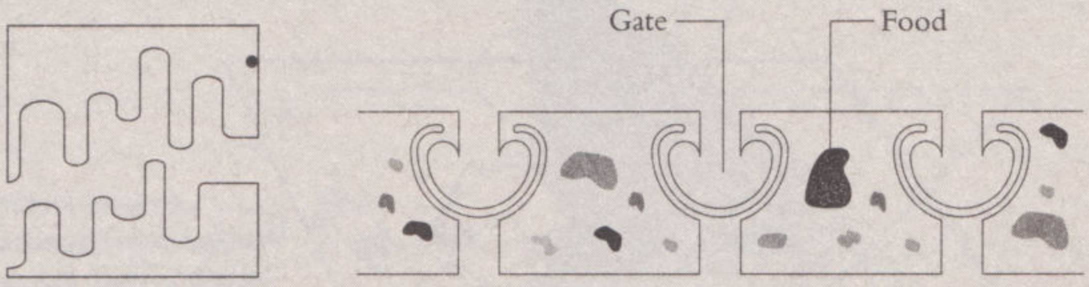
As we have already observed in Chapter 1, when a Flatlander looks at the square, or at any other Flatlander, he only sees part of its outer perimeter, and what it sees is a sort of straight line segment, and some depth in a few cases as a result of the fog. However, when the sphere observes him, he sees the complete exterior perimeter, its four sides, as well as its interior: stomach, intestine, heart and lung. If we place a one pound coin on a table and look at it with just one eye at the height of its surface, we obtain a vision similar to that of the inhabitants of Flatland when they see a priest. However, if we look at it from above the table, we see the perimeter of the coin, as well as its face.

Returning to our world, when we look at each other, we see one side of our outer surface, a flat image given perspective through depth. Three-dimensional vision is as a result of the position of our eyes (people with only one eye have flat vision). But how would we appear to a hyperbeing? According to the dimensional analogy, we can say that he would see all of our outer surface (chest, back, feet, head, etc.) and also all of our insides (heart, lungs, liver, veins, bones, etc.). And that's just one view. A doctor in the fourth dimension would be able to give us a medical check-up and discover, for example, if there is an obstruction in the valves of the heart, a kidney stone or a fracture in one of our bones, without the need for an exploratory operation, and without even radiography or ultrasound.

How is it possible that a hyperperson can see our inside and outside at the same time? The retina of a human being is like a two dimensional disc to which a number of nerve endings are connected. These carry information to the brain from our roughly spherical eye. When we look at the square in Flatland, each of its points sends a ray of light through three-dimensional space to a point on our retina, creating an image of the square. It is as if the square is transferred through space, reducing it until it reaches our eye.

THE STOMACH OF A TWO DIMENSIONAL BEING

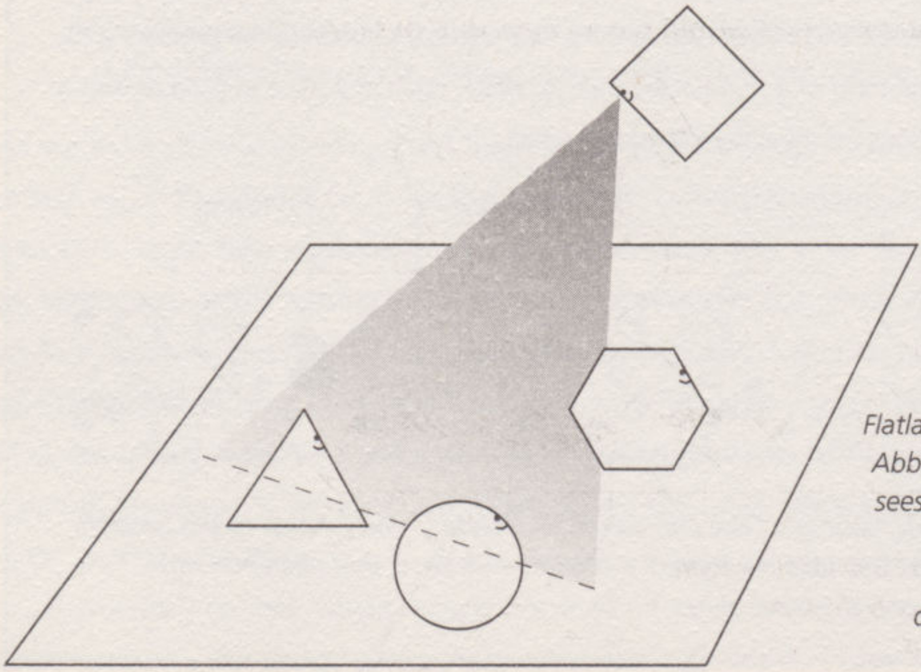
On certain occasions it is necessary to be careful with a dimensional analogy, as is the case when we imagine the digestive system of a two-dimensional being. If we think it could simply be the two-dimensional version of our digestive tract, we come up against a problem: the inhabitants of Flatland would be split into two. Obviously this would make it impossible for them to survive. The digestive apparatus of two-dimensional beings caused some interest, even controversy, at the end of the 19th century. One possible solution to the problem is for the tract to take the form of a channel with gates, which open and close as food passes: a sort of zip which stops the inhabitant breaking in half.



A square with a digestive tract like ours would be bisected. One solution to this problem would be for it to take the form of a channel with gates that open and close.

In a similar manner, the retina of a three-dimensional being could be a three-dimensional sphere with the corresponding nerve endings (the eye itself being a hypersphere). When it looks at us, each of the points of our body, outside and inside, emits a ray of light – assuming there is some kind of extra-dimensional light source – through ana or kata, to a point on this retina, creating a complete image of all of our body. It would be as if we were transferred to the retina of the hyperperson through the rays of light. An extra dimension is needed to allow the rays of light to connect each point of our body to a point on the retina of the hyperbeing.

In Abbott's novel, the square explains that he only accepted the third dimension described by the sphere when it took him out of Flatland and made him view his two-dimensional universe from Spaceland. On this trip, it is understood that the square is able, during its trans-dimensional journey, to see in the same way as the sphere. But let us go a little further. The retina of the eye of the square could be a linear segment, with its nerve endings in two dimensions. When he looks towards Flatland, his view of its two-dimensional universe will be its intersection with the spatial plane that contains the square and his view, or rather, a straight section that would pass over the flat universe as the sphere moved the square. The vision of the square would be like passing an image scanner over Flatland, although from a literary point of view, this version has less impact and would be less dramatic for the purposes of the novel.



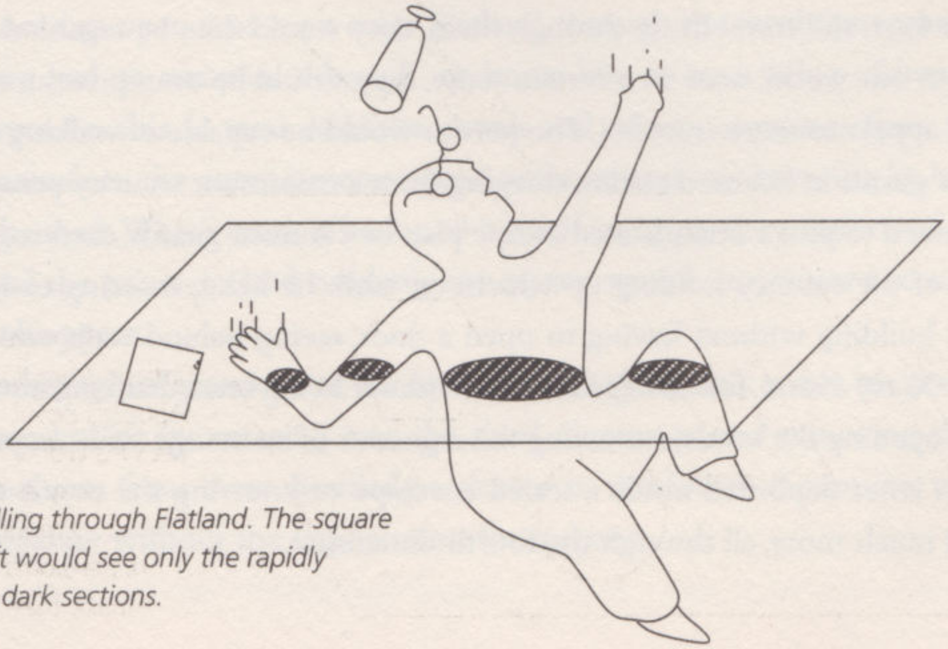
A square looking at Flatland from Spaceland. In Abbott's novel, the square sees his world in the same way as the sphere, although his two-dimensional eye would make that impossible.

Similarly, if an angel from the fourth dimension took us to heaven, looking down upon our world from ana, we would see two-dimensional sections of our universe obtained from the intersection of our three-dimensional space with the three-dimensional space that our body would represent within the fourth dimension.

A magnificent view

Another interesting question is the opposite of the one posed above: how would a hyperbeing from the fourth dimension that is visiting or crossing our space appear to us? A common opinion among those who think that spirits are beings from the fourth dimension is the belief that when we meet one of them, they will look like a being similar to us. However, the 'reality' of the matter could be quite different.

Returning to Abbott's book, when the sphere crosses Flatland, the square sees flat sections of the object, then when it is in front of him, it appears as a two-dimensional being, specifically a priest. But what would the inhabitants of this two-dimensional universe see if it were to be crossed by a human? For example, the image in Rudy Rucker's story *Message Found in a Copy of Flatland* (1983), whose protagonist falls through Flatland, which is surprisingly located one yard from the floor in the basement of a Pakistani restaurant. The Flatlanders would see a flat section of the human body, made up of various units, and the outside of these would be a deformed ring of skin and hairs, or clothing fabric. If the fingers of a hand were to pass through, they would see five small irregular discs with skin and hairs on their outside, but if it was standing up, they would see a larger disc of fabric.



Person falling through Flatland. The square on the left would see only the rapidly changing dark sections.

A hyperbeing crossing our three-dimensional universe would produce similar images. However bearing in mind the extra dimension, we would see a series of deformed blobs of different sizes, some surrounded by skin and hairs, others by fabric, and other body parts, that happen to be the sections of its hyperbody within our universe. Would that not be a terrifying sight? Certainly, we must admit that we do not know what form a hyperbeing would take, so perhaps we should allow ourselves to imagine them as similar to us, although with an extra dimension.

CATCHING A FOUR-DIMENSIONAL CREATURE

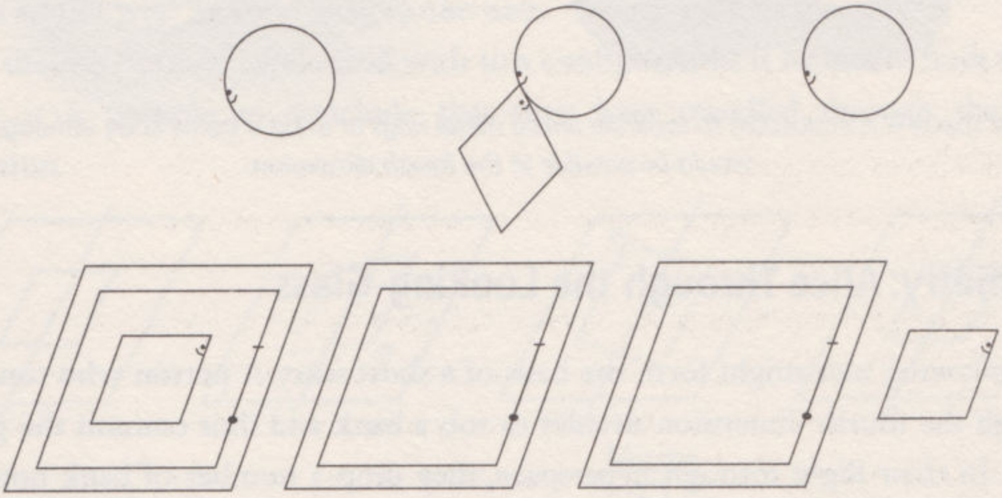
If the inhabitants of Flatland wanted to catch a human seen from their two-dimensional universe, one possible technique for doing so would be to tie a knot with a lasso around one of its sections, or some sort of metal handcuffs. If what they caught was a finger or the hips, the person could simply escape 'upward', but not if the device was placed on the wrist or the ankle.

For the protagonist in the novel *The Monster from Nowhere* (1974) by Nelson Bond, an explorer who visits the mountains in Peru and discovers a four-dimensional creature, this technique is less convincing. He traps the creature not by putting one of its spherical sections into a bag, as would be analogous to the previous method, but by obstructing it with a spear and thus leaving it trapped in our universe.

The robbery of the century

If the fourth dimension existed, and a person had the ability to pass from our world to ana or kata, and travel freely through them, they would then be regarded as a sort of deity in our world, since in a certain sense, they would be omnipresent, omnipotent and appear to work miracles. The person would be capable of walking through walls like ghosts in haunted castles, escaping from a maximum security prison without the need to plot a complicated escape plan or confront guards, carrying out the robbery of the century, holding up whichever bank he liked, entering or leaving a room or building without having to open a door, seeing behind walls without the need for X-ray vision, following someone without being seen, drinking a soft drink without opening the bottle, extracting the segments of an orange without peeling it, reading a letter deposited inside a sealed envelope or knowing the contents of any box, and much more, all through the fourth dimension.

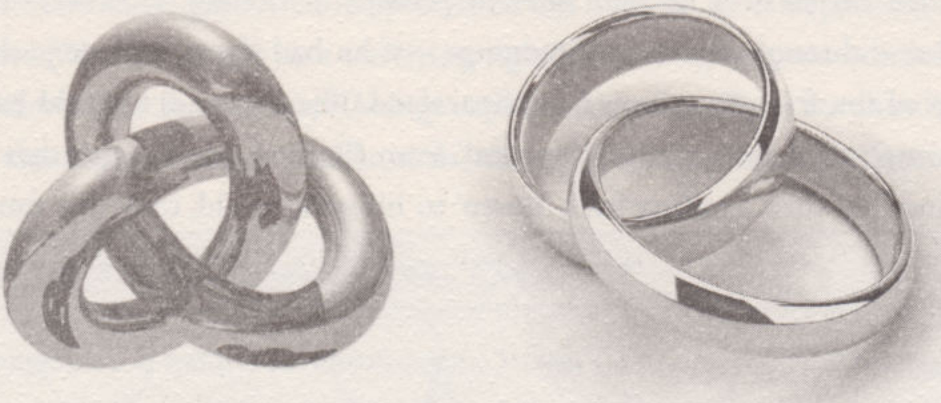
To understand how it is possible to carry out these magic tricks, we turn once again to dimensional analogy from our test bed, Flatland. What would the prison be like where the members of the Council lock up the square? It would be a square surrounded on each of its four sides to prevent it moving. However, the square could flee – through the third dimension – if he had the power or perhaps with the help of the friendly sphere from Spaceland. The escape act would be nothing more complicated than moving 'upward' from Flatland, passing through the third dimension and then heading back 'down' to his own world, only this time outside the prison.



In addition to this, the square would be able to spy on the members of the Council just by passing upward into Spaceland where he can see where they go and what they do – all without being seen. If he became thirsty during his spying, there would be no one to stop him approaching a house nearby and having a drink without the need to open it or even enter Flatland, through the third dimension. And if the square had no scruples, although we know him to be a being of integrity, he would even be able to steal valuables, jewels and other possessions from a priest's home, again from the outer dimension and without anybody knowing he had committed the theft. What's more, if he found himself enjoying the comforts of a priest's home, and the priest entered suddenly, the square would simply need to move upward to disappear.

If the hexagonal grandson of the square was choking on a sweet, his grandfather could save his life by entering into the third dimension and removing the sweet from his throat from there. Similarly, a doctor in the fourth dimension would be able to operate without the requirement for surgery.

Something that is explained in the same way, but which is often much more surprising, is that through a fourth dimension, we could separate two linked metal rings or undo a trefoil knot, as in the poem by Maxwell that opened this chapter.



In our space, it is impossible to separate linked metal rings or undo a trefoil knot, although this would be possible in the fourth dimension.

Symmetry: Alice Through the Looking-Glass

The following idea might form the basis of a short story. A person who can move through the fourth dimension decides to rob a bank and thus commit the perfect crime. In their flight through hyperspace, they drop a number of bank notes and some come to rest back in our world, where they are found by the detective assigned to the case. He observes in astonishment that they have become reversed. The detective asks himself what has happened to the notes, and what connection does this mysterious circumstance have with the hold up at the bank?

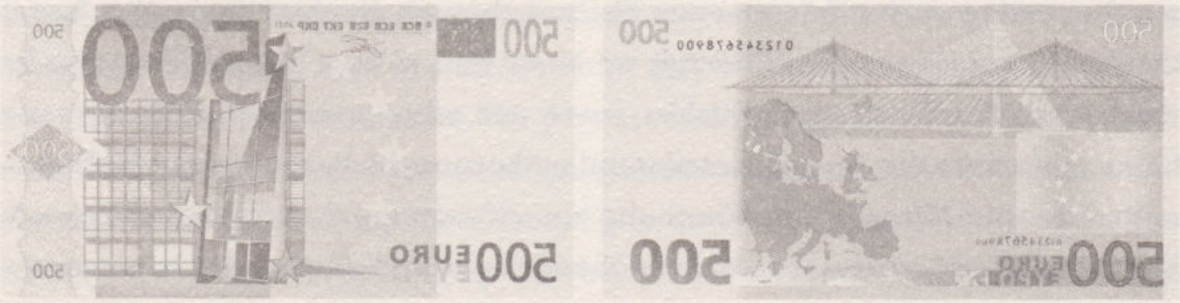


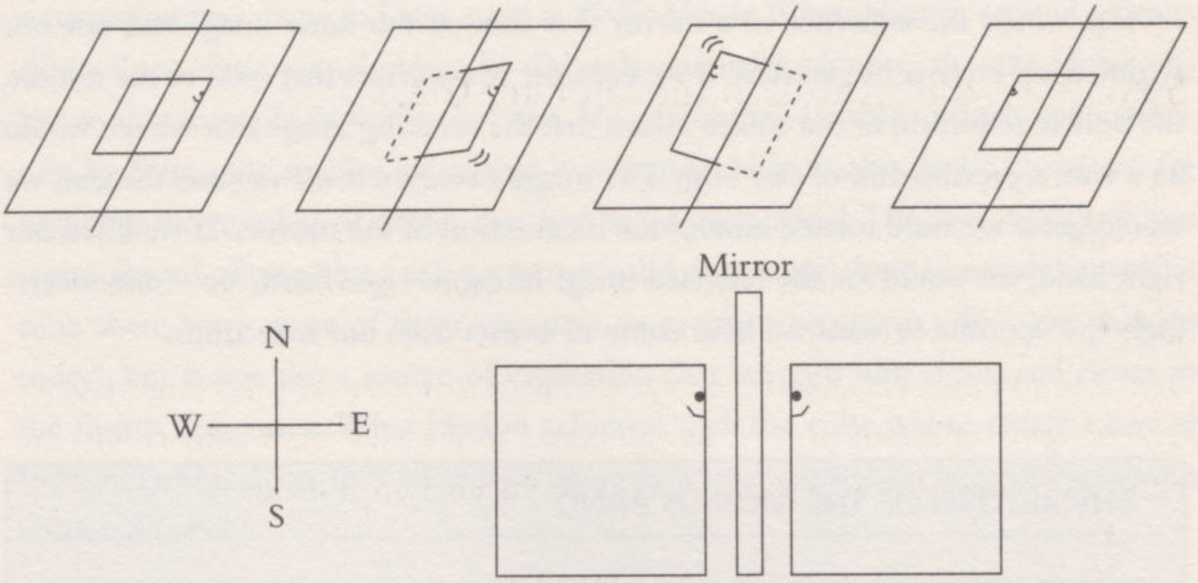
Image of the two sides of a bank note which have travelled to the fourth dimension and returned flipped around.

In short, how can we know if a person has managed to travel through the fourth dimension, or if a Flatlander travelled through Spaceland?

Let us return to Flatland once more. If we rotate our square friend around one of the lines that cross him, as shown in the following illustration – that is to say, removing him from the plane and rotating him in the third dimension – we obtain his image "through the looking-glass".

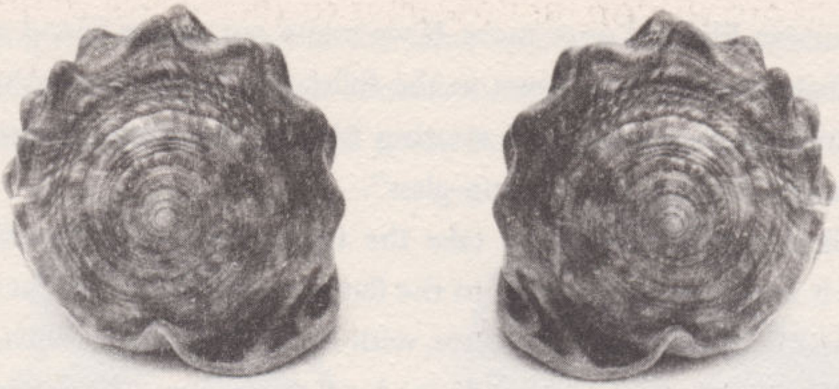
The result is the same as if we take the square from the two-dimensional world, rotate it in space and return it to the flat universe. Suppose that any inhabitant of Flatland, and thus also the square, with its head pointing north, has its eye and mouth towards the east and inside, its lung to the west. If we were then to rotate the square on an axis, we would obtain its mirror image. The eye and mouth to the west and the lung to the east.

As there is nobody in Flatland with this configuration, if someone finds such a square, it is possible to conclude that they have travelled through the third dimension.



Square rotated in the third dimension and its resulting mirror image.

Now suppose that we are rotated in four-dimensional space around a plane that crosses us from top to bottom (note that the rotation now occurs around a plane and not a straight line), or that we are transferred to hyperspace and rotated there. The result would be that we would remain ourselves, but be flipped. What was once on the left, such as the heart, would now be on the right. In fact, if a body was rotated in the fourth dimension, we would be changing its orientation. For example, in the case of a snail whose shell turns in a clockwise direction, this would now turn in the opposite direction. The same would happen to any right-handed object, being transformed into a left-handed one.



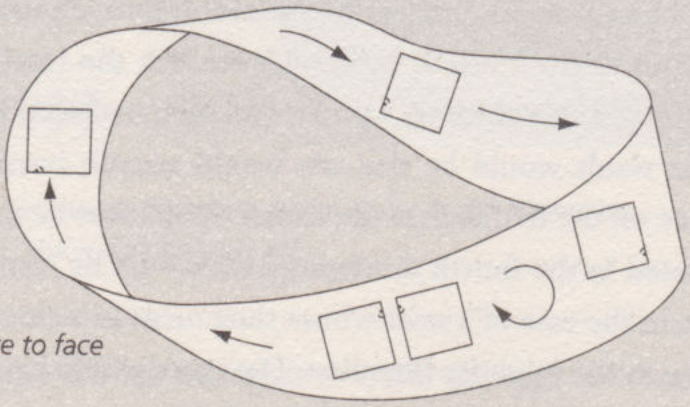
Snail with anti-clockwise spiral, and its image after having being rotated in the fourth dimension.

When we look in the mirror, the image that we see is of the 'person' who would return to our world if we were rotated in the fourth dimension. If we lift our right hand, the image on the other side of the mirror lifts its left.

But would the existence of a mirror that showed our same image and not our symmetrical brother be possible? If we consider two mirrors that cross in the middle, the double reflection of our image means that the resulting image that we see would be a true representation of our body. The image obtained is the same as the one we would get if we were rotated around the intersection of the mirrors. If we lifted our right hand, we would see the reflected image lifting its right hand, too – disconcertingly the opposite to what we have come to expect from our reflections.

THE RETURN OF THE MÖBIUS BAND

If the universe in which the square and the other Flatlanders lived was shaped like a Möbius band, which is a single-sided shape, it would be possible for a square and its mirror image to be face to face, contrary to what would happen in a flat or any other kind of universe. The same would happen for any other dimension, including our three-dimensional universe.



A square and its mirror image, face to face on a Möbius band.

Charles Howard Hinton, philosopher of the fourth dimension

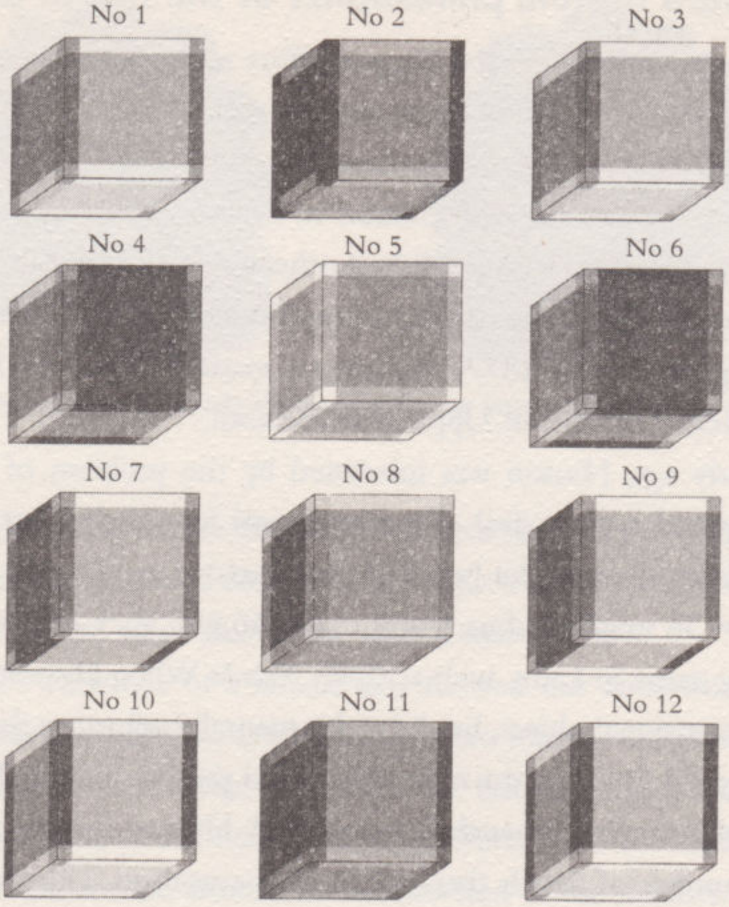
One of the figures who made the biggest effort to understand and promote the understanding of the fourth dimension was the mathematician Charles H. Hinton. He was interested in topics from mathematics and physics, as well as philosophical and religious matters, and the visualisation of four-dimensional space, particularly the hypercube, although he also published other surprising and speculative aspects.

Charles Howard Hinton was born in London in 1853. He studied mathematics at Oxford and graduated in 1877, obtaining a masters degree in 1886. He started working as a science teacher in Uppingham School.

From an early age, Hinton was interested by the problem of visualisation. At Oxford, he acquired a great deal of mathematical knowledge, but he did not feel fully satisfied. From then on, he began to work with a cubic yard (91.5 cm), made up of 36 × 36 × 36 smaller cubes, that is to say, 46,656, each of which was assigned a corresponding name in Latin, such as *Collis Nebula*. When Hinton wanted to visualise a three-dimensional object, he did so by mentally adjusting the size of this object so that it would fit inside the cube. He would then be able to discover its structure by listing the smaller cubes that it occupied. Hinton also devised a system for reducing the number of details that had to be memorised. The idea, which at first seems absurd, allowed him to have a true 'solid document' (bear in mind that at that time there were none of the computers or graphics programs which we rely on today), but it was also a source of inspiration that brought him closer and closer to the fourth dimension. What Hinton achieved with the cube was to create a sort of four-dimensional eye that would go on to form the inspiration behind his famous coloured cubes.

JAMES HINTON

The young Charles Hinton was strongly influenced by a group of intellectuals with progressive social and political ideas. They included Havelock Ellis, doctor and sexologist, George Boole, founder of mathematical logic, and his wife, the mathematician Mary Everest Boole. The most radical of them all however, was Charles's father, James Hilton, who had a career as a surgeon before becoming a famous writer, philosopher and champion of 'a new sexual morality'. In his final years, Hinton senior had a circle of admirers all of whom he maintained sexual relationships. One of his favourite sayings was "Jesus Christ was the saviour of man, but I am the saviour of woman. And I don't envy him in the slightest".



Hinton's coloured cubes are a complicated set of 12 cubes with coloured faces, edges and vertices, which according to Hinton made it possible to visualise a hypercube. A total of 81 colours are used, with names in Latin that correspond to the 81 parts of the hypercube which has 16 vertices, 32 corners, 24 faces, 8 hyperfaces and 1 hypercube. Hinton's cubes enjoyed great fame. They were published in women's magazines and were even used in seances. As objects, they took on an almost mystical symbolism, since they could allegedly be used to see ghosts and dead relatives from the fourth dimension.

Hinton's interest in the fourth dimension continued to grow, and in 1880, he published the article "What is the fourth dimension?" in the Dublin University Magazine, which was republished in 1883 in the Cheltenham Ladies College Journal. The following year, it appeared as a pamphlet entitled "Ghosts Explained", published by Swan Sonnenschein & Co., a company that published nine pamphlets, essays on the fourth dimension and science-fiction stories that would be later collected together under the title *Scientific Romances*. Among these was *A Plane World* (1884), a similar idea to that of Abbott's *Flatland*, although Hinton was more interested in the physical aspects of this two-dimensional world, which he conceived on a sphere instead of a plane.

THE BASEBALL GUN

During his period at Princeton, Hinton dedicated a lot of time to the design of the 'baseball gun', a machine that pitched balls to batsmen, allowing them to train without the need to wear out pitchers. The machine expelled baseballs at a speed of between 60 and 112 km/h. However, despite being used for a number of years, the machine eventually fell out of use – it was just too dangerous to use. Nevertheless, to this day gadgets derived from Hinton's invention are still used on baseball fields.

Hinton was able to live a comfortable life and achieved a moderate degree of social success. Then in 1885, things soured. He was arrested for bigamy. He lost his job, his career was ruined and, following the sentence – three days in jail – he moved to Japan with his family where he worked as a secondary school teacher in Yokohama. In his exile, he sent the manuscript for *A New Era of Thought* to friends who published it in 1888. The first part of the work is dedicated to the issue of thinking "four-dimensionally" and philosophical and religious aspects related to the fourth dimension. The second part is focused on visualising the hypercube, and in it we find the description of the coloured cubes and their instructions for use.

Towards 1893, Hinton began his tour of North America, finding work at the universities of Princeton, Minnesota and later on Washington, D.C., as well as the U.S. Naval Observatory and the Patent Office. He made the fourth dimension popular in the United States and became a recognised and respected individual in the most intellectual circles. Hinton wrote many articles and gave many lectures on a range of subjects, including poetry. In 1904, he published the book *The Fourth Dimension*, which brought together his reflections on the subject, as well as a novel set in a two-dimensional universe, *An Episode of Flatland*. He died in 1907.

Chapter 5

Gods and ghosts

> *But because we do not hear the high or low sounds, or see the colour produced by rapid or slow colour vibration, it does not follow that they do not exist. Is it not more than possible, is it not more than probable, that there is a fourth dimension to which our eyes have not been opened, and that our so-called dead are living in this world, and that through our own development communication with them will come; that this new world is all around and about us, and is a world of an infinite variety of colour and sound?*
>
> *Charles B Paterson,* A New Heaven and a New Earth, or the Way to the Eternal Life *(1909)*

The fourth dimension had all the necessary ingredients to ensure that at the end of the 19th century and the start of the 20th it would attract the attention of the universe of beliefs: conventional religions, new religious movements, sects, the world of the paranormal, the occult, philosophy, theology, spiritualism, mysticism... It grew to become a highly significant topic in the religious world and we can see this in the books and articles published at the time. However, if we carry out a search of Internet pages and books today, we will discover that the fourth dimension continues to be a topic that captivates.

Spiritualism and ghosts in the fourth dimension

Spiritualism, the belief that the spirits of the dead are close by and willing to make contact with those who invoke them, arose in Europe as a religious and philosophical movement in the 19th century. It became strongly embedded in the United States soon after, giving rise to a growing deluge of reports of paranormal phenomena. At that time, a large number of mediums, many from the United States, organised sessions to contact the spirits, mixing religious beliefs, mysticism and the feelings of those who came to them to speak to their loved ones, in the spectacle staged by these messengers from beyond. The work of mediums lay more in psychology or prior research than in contacting spirits, and was accompanied by magic tricks and elements of theatre. Mediums freqently found themselves under suspicion of fraud, since their world was filled with colourful anecdotes and paraphernalia but completely lacking in scientific research.

Only a few scientists were interested in the spirit world, but there were some, as we shall see further on, who attempted to prove its existence. One of the most distinguished defenders of scientific spiritualism was the English chemist William Crookes (1832-1919), the inventor of the cathode ray tube, which formed the basis of the first television sets and computer monitors.

In terms of the nature of the spirits themselves, there were two trends. The first, more popular among spiritualists, maintained that they were immaterial three-dimensional beings composed of energy, ectoplasm or another type of supernatural substance. But if they were immaterial, how could they move objects during seances? On the other hand, a stronger trend by the end of the 19th century argued that spirits were material beings and the fact that we could not see them was because they existed outside our space and visited us at their whim. They were, for example, beings that inhabited the fourth dimension. The materialisation of spirits,

The 19th century saw a proliferation of mediums, people who claimed to have powers that allowed them to make contact with entities from the fourth dimension. One of the scientists who defended spiritualism was the Englishman William Crookes (above). The photograph on the left shows the spectre of Mr Maskelyne appearing before Dr Weatherly.

which became such a popular subject at the time, was nothing more than their presence in our three-dimensional space. Some spiritualists criticised this materialist trend, arguing that if spirits were material, they would not be able to pass through doors or walls. However, for ghosts in hyperspace, this would be possible by passing through the fourth dimension, as has been explained in the previous chapter.

SIR WILLIAM CROOKES' SCIENTIFIC SPIRITUALISM

William Crookes before the spirit of Katie King.

The English chemist, who also worked in physics, was one of the most important scientists in Europe at the time. His work included the invention of the cathode ray tube, the study of the conduction of electricity, the discovery of thallium, the development of the amalgamation process for the separation of silver and gold from other minerals, the invention of chemical dyes for the textile industry, and research into obtaining industrial diamonds. Moreover, in addition to all this, Crookes was a pioneer of research into psychic phenomena, and held the position of president of the Society for Psychical Research. In 1870 he wrote one of his best known articles, *Spiritualism Viewed in the Light of Modern Science.* Crookes studied the materialisation of spirits and the work of a number of famous mediums, such as Daniel D. Home, Katie Fox and Florence Cook. The last of these was a young lady from London with the ability to invoke spirits, which spoke or materialised in her presence. Her most famous materialisation was that of the spirit of Katie King, the daughter of the pirate Henry Morgan. Crookes managed to take 44 photographs of Katie, as well as taking her pulse and cutting off a lock of her hair. It is said that the scientist ended up falling in love with the ghost. All of this, which was published in his book *Researches in the Phenomena of Spiritualism*, constituted a scandal that was further exacerbated by the arrest of a woman whose looked identical to the spirit of Katie King.

The people who made the idea of the existence of spirits as beings from the fourth dimension most popular were the American medium Henry Slade and the German physicist Johann Zöllner. As we have briefly noted already, the fourth dimension acquired a certain degree of notoriety as a result of Slade being tried for fraud in London. He had been invited by Coronel Olcott and Madame Blavatsky, founders of the Theosophical Society, based on the interest aroused in the Grand Duke Konstantin of Russia by his research into spiritualism. Seances organised by Slade soon became extremely popular among those who moved in spiritualist circles and London's upper class. However, some scientists ended up accusing the medium of fraud. During one session, they observed that one of the slates commonly used for the spirits to write their messages on during meetings had already been written on before it started. The judge sentenced him to three months of hard labour. The sentence was eventually overturned on a technicality, and Slade left England.

The case of Slade became famous in the media, and details were raked over by the newspapers, becoming a hot topic. The case caused scandal in English high society, and although there had been others related to spiritualism, it was the case of Slade that became most famous, as prestigious scientists throughout the world came to the defence of the medium. These included Johann Zöllner; William Crookes; the German physicist Wilhelm Weber (1804-1891), a colleague of Gauss and mentor to Riemann; the English physicist Joseph J. Thomson (1856-1940), soon to be a Nobel laureate for his discovery of the electron; and the English physicist Lord Rayleigh (1842-1919), also a future Nobel laureate for his research

A CINEMATIC MEDIUM

Mediums have been widely represented in the cinema in films such as *Seance on a Wet Afternoon* (Bryan Forbes, 1964), *Juliet of the Spirits* (Federico Fellini, 1965), *The Changeling* (Peter Medak, 1979), *Poltergeist* (Tobe Hooper, 1982), *Ghost* (Jerry Zucker, 1990) and *The Orphanage* (Juan Antonio Bayona, 2007). However, one of the most notable fictional psychics is the protagonist of the film *Family Plot* (Alfred Hitchcock, 1976). The loveable Blanche is a clairvoyant who sweet talks and fascinates the old ladies who request her services, extracting information from them while making contact with the ghost of a dead relative, all the time accompanied by lights and sound effects. Meanwhile, her taxi-driver boyfriend, lends a hand by investigating the backgrounds of the wealthy patrons.

In 1875, Helena Blavatsky, also known as Madame Blavatsky, co-founded the Theosophical Society based in New York.

into the density of numerous gases and the discovery of argon. These luminaries testified that spirits existed and that the paranormal phenomena for which Slade was responsible were possible as a result of space being four dimensional. Ghosts, they said, were beings who lived in the fourth dimension.

A year after having taken flight from London, Henry Slade reappeared in Leipzig at the invite of Zöllner who, together with a number of colleagues including Weber and Fechner (the author of the story *Space Has Four Dimensions*), hoped to carry out a series of experiments that would prove once and for all that spirits were four-dimensional beings, and thus, the existence of a fourth spacial dimension. Zöllner was familiar with multidimensional spaces from his physics research and knew the work of Gauss, Riemann and Helmholtz, realising that it could be used to explain paranormal phenomena.

After a number of months of seances held by the Leipzig group, Zöllner published two works in London. In 1878 the article *On Space of Four Dimensions*, and in 1880, the translation of the third book of his work *Wissenschaftliche Abhandlungen* retitled *Transcendental Physics*, were a compendium of the experiments. The book enjoyed great popularity, becoming a standard reference for many of those with an interest in spirits, such as theosophists or certain artists, including the Russian expressionist painter, Wassily Kandinsky.

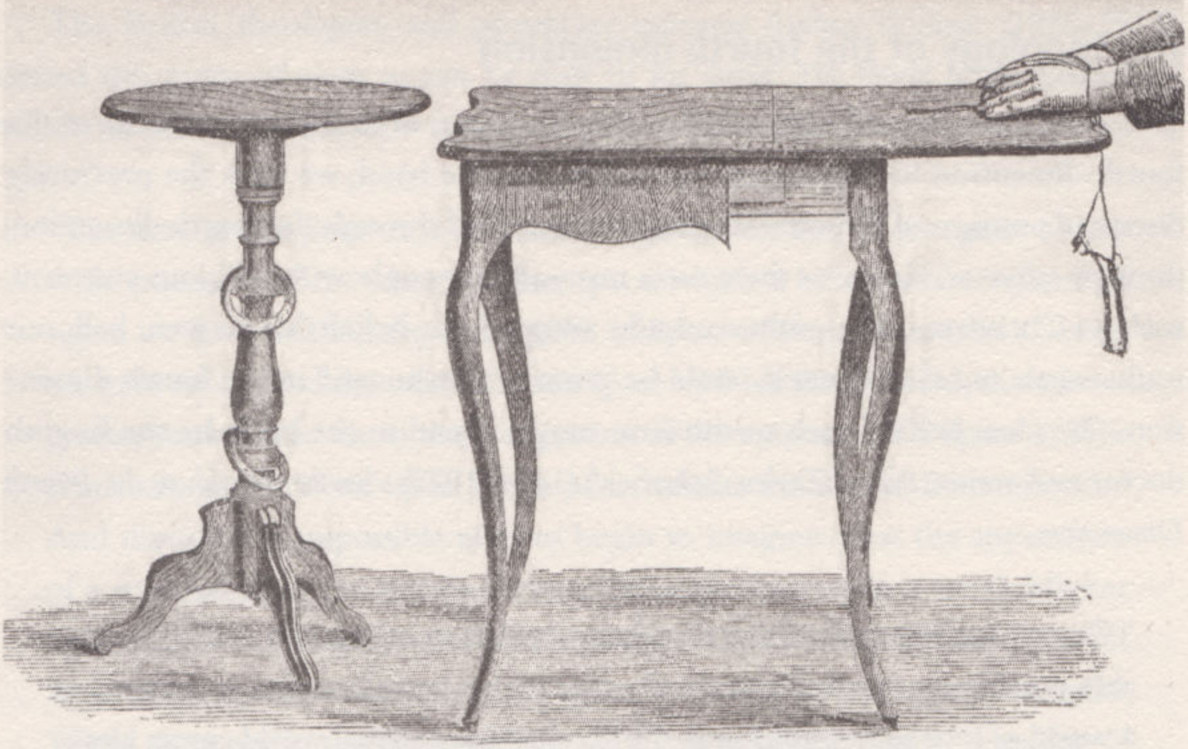
The first step taken by the American medium was to take a rope and tie the ends together, thus making a closed loop. After Slade had placed his hand on the rope, four simple knots would appear. Given a closed loop, it is impossible to make a knot in it from within three-dimensional space, except by undoing the rope, making the knot and then tying it again. However, it would be possible for a spirit from the fourth dimension to do so, although to tie the knot, it would need to move the rope to ana or kata. For Zöllner, the alleged success of this phenomenon proved the existence of spirits from the fourth dimension.

The book *Transcendental Physics* contains details of many of the paranormal phenomena executed by Slade at the meetings of the Leipzig group, in addition to a series of experiments personally designed by Zöllner to prove the four-dimensional nature of spirits. For example:

1. One of the experiments consisted of the spirits linking two wooden rings without breaking them, through the fourth dimension.
2. Natural elements often have the property whereby one of their parts has a given orientation, such as snails. The reflectional symmetry resulting from a trip to the fourth dimension would change this orientation.
3. A length of catgut is used to make a closed loop, along which a knot must be made.

But were Zöllner and Slade's experiments truly successful? Zöllner thought so, but in terms of the scientific method, the experiments themselves were flawed. The spirits did not do what Zöllner had expected according to the original design of his experiments. Instead they put some rings around the leg of a pedestal, the snail passed from a table to the floor and the cat gut ended up with holes in it.

Few people were satisfied by Zöllner's explanations, and this caused fierce debate on the issue among intellectuals, with scientists of the stature of Helmholtz criticising him strongly. Detractors of the spiritualist physicist believed that a



An image from Zöllner's book Transcendental Physics *which shows a pedestal table with two wooden rings around its leg, supposedly placed there by spirits from the fourth dimension, although the spirits were meant to have linked up the rings. In theory, these are also responsible for the knots tied on the rope shown on the right.*

scientist is not the best qualified person to evaluate the practices of a magician, since while he is watching his right hand, the left hand is carrying out the trick. The prevailing opinion in academic circles was that Zöllner had allowed himself to be tricked by Slade and was perhaps unhinged.

The effect of Zöllner's work was to make the fourth dimension a joke, far removed from any scientific reasoning. However, the idea that spirits are beings from the fourth dimension was taken up again at the end of the 19th century by English protestant ministers, such as Edwin Abbott, who distanced themselves from mediums and made use of the concept for more profound theological debates. In addition to this, figures such as Hinton continued to work on the more serious aspects of the fourth dimension.

The theology of the fourth dimension

Two stances can be easily identified as simplifications of those that made use of the fourth dimension in theological issues. On the one hand, we have the previously discussed position of Abbott: "We cannot reach God through the fourth dimension, through science." However there were many other people with religious concerns, such as Christians, who enthusiastically adopted the belief that heaven, hell, our souls, angels and God himself, could be 'comfortably housed' in the fourth dimension. The ideas behind such spiritualism can be found in the book by the English doctor and writer Alfred Taylor Schofield (1846-1929) *Another World, or the Fourth Dimension*:

> "We conclude therefore, that a higher world than ours is not only conceivably possible, but probable; secondly, that such a world may be considered as a world of four dimensions; and, thirdly, that the spiritual world agrees largely in its mysterious laws, in its language that is foolishness to us, in its miraculous appearances and interpositions, in its high and lofty claims of omniscience, omnividence, etc., and in other particulars with what by analogy would be the laws, language and claims of a fourth dimension...
> ...though the glorious material universe extends beyond the utmost limits of our vision, even artificially aided by the most powerful telescopes, that does not prevent the spiritual world and its beings, and heaven and hell being by our very side..."

Two brief remarks on Schofield's vision. Contrary to conventional wisdom, and the English doctor's idea, if angels or spirits could pass through our world as four-dimensional beings, this would not lead *a priori* to a single entity, nor to them having a similar appearance to humans, as we have shown in Chapter 4.

Moreover, why would God, in his perfection, have specifically chosen the fourth dimension in which to exist? Why not the fifth or the sixth, or any other higher dimension? A two-dimensional plane is found within a three-dimensional space that is also within another four-dimensional space, and so on, up to an infinite number of dimensions. For the existence of a perfect being such as God, who is all-powerful and all-seeing, perhaps we should think about a space with infinite dimensions. Philosophers of the fourth dimension had already made similar reflections during the 19th century.

The British theologian and protestant minister Arthur Willink (1850-1913) shared this point of view, as can be seen in his work *The World of the Unseen*, in which he maintains that God inhabits a space with infinite dimensions:

> "But now at this point it is necessary to go farther and to recognise a very wide extension of the idea of higher space, which is by no means exhausted when we have reached the conception of Space of Four Directions ... when we have recognised the existence of a space of Four Dimensions, there is no greater strain called for in the recognition of the existence of a Space of Five Dimensions, and so on up to Space of an infinite number of Dimensions... And though it is impossible even to begin to imagine what the appearance of a material object in our Space may be to an observer in a much Higher Space, still it is evident that to him is presented a still more infinitely perfect view of its constituents than to an observer in any Lower region of Space. While to an eye in the Highest Space of all, an infinitely perfect revealing of the most hidden and secret things is of necessity presented.
>
> This emphasises very strongly what has been said about the Omniscience of God. For He, dwelling in the Highest Space of all, not only has this perfect view of all the constituents of our being, but also is most infinitely near to every point and particle of our whole constitution. So that in the most strictly physical sense it is true that in Him we live and move and have our being."

THE ALGEBRA OF HUMAN BEINGS

Alfred T. Schofield made use of an algebraic metaphor to discuss the material and spiritual parts of human beings:

"Another curiously universal, instinctive belief and one by no means confined to Christianity, is, that when a man dies, part of him (his soul, or spirit) leaves this world altogether, to enter the higher one. And here we can remark that the general belief that man has a spiritual nature...may be well illustrated by algebra. Let, for example, the body, material and solid, be represented fairly enough by x^3, and the spirit, higher and possessing an unknown power, by x^4. Then, $(x^3 + x^4)$ represents the man in life, while $(x^3 + x^4) - x^4$ represents the departure of the spirit (x^4) at death, which returns to its own dimension, while the bodily remains (x^3) return to the earth to which it belongs."

At the same time, the German mathematicians Richard Dedekind (1831-1916) and, above all, Georg Cantor (1845-1918) were carrying out their mathematical studies of the infinite, perhaps for the first time in history with certain rigour. Subsequently, at the start of the 20th century, the German mathematician David Hilbert (1862-1943) introduced spaces with infinite dimensions in which it was possible to measure distances, so-called Hilbert spaces.

The philosopher and mathematician William A Granville (1864-1943), author of the article *The Fourth Dimension and the Bible*, shared the belief that God inhabited a space with infinite dimensions. However, he thought that the fourth dimension and other higher dimensions were heaven and the second and lower ones, hell. Thus when a person dies, they are limited to the upper dimensions if they deserve to go to heaven, otherwise they pass down to the lower ones.

HILBERT'S INFINITE HOTEL

Imagine a normal hotel, with a finite number of rooms, all of which are occupied. If a traveller were to arrive and request a room, the owner would have to reply that it is not possible for him to stay. However, let us assume that the hotel has an infinite number of rooms, numbered according to the natural numbers, 1, 2, 3, 4..., and that, as in the previous case, it is full. A new guest arrives and asks for a room. "Certainly", the owner replies, "here's what we'll do: the guest in room 1 will move to room 2, the guest in room 2 to room 3, the guest in 3 to 4, and so on. This means that room 1 will be free, and the new guest can stay there".

But what happens if an infinite number of customers arrive? Here there is also a solution. The guest in room 1 moves to 2, the guest in room 2 to 4, from 3 to 6..., meaning that each guest moves to the room whose number is double the number of their previous room. Thus, the rooms with odd numbers become free and the new customers can stay there.

The concept of infinity contradicts certain 'truths' that seem logical to us and which obviously hold for finite sets. Consider, for example, the statement that the "the sum is greater than the part". This is not true for infinite sets. As is shown by the infinite hotel, the set of even numbers has the same number of elements as the set of natural numbers.

Mysticism, theosophy and the astral universe

The Russian philosopher and writer Piotr D. Ouspensky (1878-1947) remarks in his essay *The Fourth Dimension* that, contrary to what we believe, perhaps we are not three-dimensional beings. For him, belief in the existence of a fourth dimension inevitably leads to one of two consequences: Either we are four-dimensional beings, or we only have three dimensions. However, the latter would mean that we would not physically exist.

If there were a fourth dimension and we were three-dimensional beings, this would mean that we did not have a real physical existence: We would be notional beings not made from matter, like a point that did not have length on a straight line, a straight line which did not have width on the plane, or a plane which did not have volume in three-dimensional space. As such, we would only exist in the mind of a supreme being, whether we called him God or something else, and all our acts, thoughts and feelings would be nothing more than a product of the mind of that being, in which our artificial existence takes place at the whims of his imagination.

If we do not believe that we belong to a world that depends on a higher being and its whims, we must then have a four-dimensional reality. That is to say, it is not just spirits or ghosts but also ourselves that are four-dimensional beings. However, only part of our being inhabits the observable three-dimensional universe, and we are only specifically aware of this part of our body and our being. This reflects the notions of Plato's Cave.

Piotr D Ouspensky was a key figure behind esoteric philosophy. For him, we exist as nothing more than part of the mind of a higher being.

For Hinton and Ouspensky, the fourth dimension is not only a conceptual space, but also a sort of knowledge, an awareness of a higher reality. For them, the mathematical study of the fourth dimension naturally involved mystical beliefs, which can be simply summarised as: All is One, and the One is unknowable.

The mystical 'One' is the extent to which we can reach universal unity. It is the 'superspace' that brings everything (the near and the far, the past and the future, the real and the imaginary) together in one great unity (the unity referred to by mysticism), which to mathematicians is hyperspace and which to others is God, the Absolute, the All, or whatever we wish to call it, but which cannot be represented using human symbols. This brings us to the second of the statements: "the One is unknowable". But what does this way of thinking say to us? From the point of view of mysticism, we can understand and know what the One is, in the sense that we can feel space around us or can open our hearts to feel life, beauty, love. However, it is only unknowable to the rational mind.

Rudy Rucker, in *The Fourth Dimension* (1984), makes use of the following analogy to explain this. Consider an infinite set such as the natural numbers, $\mathbb{N} = \{1, 2, 3, 4\ldots\}$. Given the idea of number, we can understand that it is $\mathbb{N}$, but if we insist on complete knowledge, that is to say, the list of all natural numbers, $\mathbb{N}$ becomes unreachable, unknowable.

In addition to, this theosophists generally have a strong interest in the fourth dimension, in spite of the fact that the founder of the Theosophical Society, Madame Blavatsky, had no inclination towards it herself. Together with philosophers of the fourth dimension, Hinton and Ouspensky, they share the belief in the unity of mysticism as well as believing in the occult, and there is a strong connection between theosophy and spiritualism. In addition to this, many theosophists, such as the minister of the Anglican church, Charles W. Leadbeater (1854-1934), believed that the fourth dimension was the astral universe, parallel to our visible universe, and that the idea of astral vision was perfectly explained by the fourth dimension: "…for astral sight the theory of the fourth dimension gives a neater and more complete explanation".

Chapter 6

The fourth dimension in literature

> *What might time have dreamt until now, which is, like all nows, the apex?… It has dreamt of space. It has dreamt of music, which has no need for space. It has dreamt of the art of words, even more unexplainable than music, as it encompasses music. It has dreamt of a fourth dimension and the strange fauna that inhabit it. It has dreamt of the number of grains of sand. It has dreamt of transfinite numbers, which it is still counting.*
>
> *Jorge Luis Borges,* Someone is Dreaming

Hinton and Abbott's work on the fourth dimension were happily picked up by writers of the time, both those of the emerging science fiction, among them H.G. Wells, and those of other literary genres. This chapter is divided into two parts, reflecting the interest shown by literature in the fourth dimension: the golden age, which goes from the end of the 19th century to the 1920s, and the time since then, after the popularisation of the theory of relativity, when interest in the spatial fourth dimension began to fade.

The golden age

We are going to begin this journey through the fourth dimension's influence on literature with the man who could be considered, along with Jules Verne, the pioneer of the literary genre of science fiction, Englishman H.G. Wells. The father of science fiction spent his youth and part of his adulthood in the golden age of the fourth dimension, therefore it is not a coincidence that his work is deeply influenced by the ideas around multidimensional spaces: journeys to the fourth dimension, visits from superior life forms, parallel universes and time machines, among others.

His best-known novel could be *The Time Machine* (1895), based on his first short story, *The Chronic Argonauts* (1888). In it, Wells considers time to be the fourth dimension, not in the relativity sense (which would become popular from the 1920s), but in the sense of static space-time, as it was regarded in that period. Hinton and other fourth dimension philosophers also believed that time could be considered as another dimension, forming a space-time continuum together with the other three spatial dimensions, through which we could travel forwards and backwards in time at various speeds before stopping at a specific moment or point in time. However, we always perceive the passing of time in one direction and at a constant speed. Likewise, at the time, movements in time was one way of thinking about or visualising the fourth dimension.

This is the static space-time that Wells embodies in his novel. The protagonist, a scientist who has been working on the geometry of the fourth dimension, builds a machine that can travel through the dimension of time. He travels to the future, to

HERBERT GEORGE WELLS (1866-1946)

His name is associated with great science fiction titles: *The Time Machine, The Invisible Man, The War of the Worlds* and *The Island of Dr. Moreau*. His family went through much economic hardship meaning he had to combine his studies with basic jobs. As a child he had to spend several months in bed as the result of an accident and it was at this time that he became a book lover, which encouraged him to write. He received a grant to study biology at the Normal School of Sciences in London. Later, Wells worked as a professor and collaborated on newspapers and magazines, but his economic problems continued. After suffering from tuberculosis which left him 40kg lighter, he decided to abandon everything in order to dedicate himself to writing. In 1895 he published *The Time Machine*, which was an immediate hit and made him a famous – and wealthy – writer, never

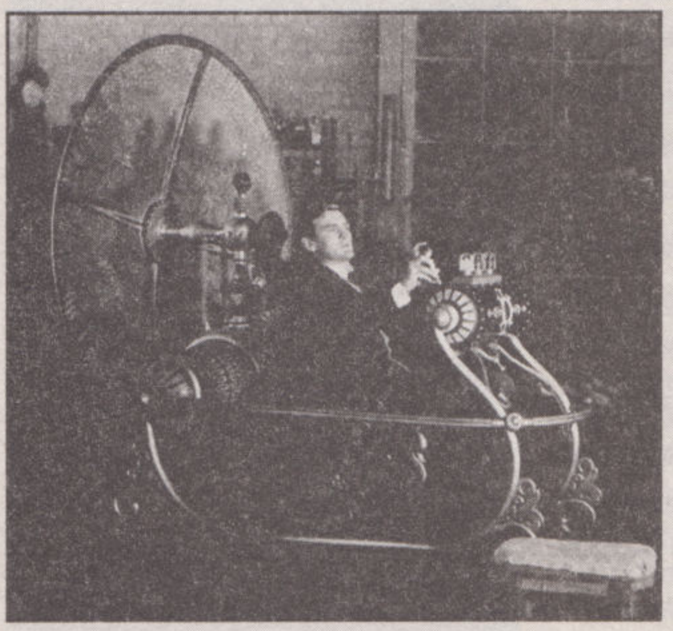
Wells was a science-fiction pioneer. Several films were made of his stories, among them The Time Machine, *a still from which appears to the right together with a Spanish poster for a cinema version of* The Island of Dr. Moreau.

the year 802701, and can see how life and society is in that age. The novel combines the author's worries about the future of humanity and reflections on the class divisions of the late-Victorian era.

The following is a quote taken from the part of the novel where the fourth dimension is mentioned:

> "Clearly," the Time Traveller proceeded, "any real body must have extension in four directions. It must have Length, Breadth, Thickness – and Duration. But through a natural infirmity of the flesh, which I will explain to you in a moment, we incline to overlook this fact. There are really four dimensions, three which we call the three planes of Space, and a fourth, Time. There is, however, a tendency to draw an unreal distinction between the former three dimensions and the latter, because it happens that our consciousness moves

to face economic hardship again. He wrote more than 100 books. A liberal person throughout his whole life, he criticised the stiffness and rigidity of Victorian society, defended the rights of less fortunate people, supported the suffragette movement, and believed in science and education as tools for improving human life. However, in his opinion, the ethical boundaries of science needed monitoring and technology should not be blindly trusted.



Burt Lancaster
Michael York
LA ISLA DEL DOCTOR MOREAU
The Doctor Is In - SANE!

> intermittently in one direction along the latter from the beginning to the end of our lives....
>
> "Some philosophical people have been asking why three dimensions particularly – why not another direction at right angles to the other three? – and have even tried to construct a Four-Dimensional geometry. Professor Simon Newcomb was expounding this to the New York Mathematical Society only a month or so ago. You know how we can represent a three-dimensional solid on a flat surface, which has only two dimensions, and similarly they think that by models of three dimensions they could represent one of four if they could master the perspective of the thing. See?"

His next novel, *The Wonderful Visit*, also published in 1895, is based on the idea of three-dimensional adjoining parallel universes within the fourth dimension and a visit from superior beings. An angel falls from its celestial realm, which he calls the world of dreams, to our world, which according to the angel "are as close as two pages in a book". Specifically, it falls into a village in England. The angel is conscious of the four-dimensional reality of the cosmos and plays the role of the sphere from *Planeland,* while the inhabitants of the English village represent the square. When the angel spoke to the vicar from his world of dreams, the vicar replied: "It is confusing, it almost makes one think there may be Four Dimensions after all. In which case, of course … there may be any number of three-dimensional universes packed side by side…"The same idea is used in Wells' book *Men Like Gods* (1923).

Another subject related to multidimensional geometry can be found in the story *The Remarkable Case of Davidson's Eyes* (1895), whose protagonist has an accident in the laboratory which apparently leaves him blind, as he cannot see anything around him. However, he does see the sea, a beach, sand, rocks and penguins. In this tale Wells uses the notion of the warping of space in the fourth dimension allowing Davidson to see events on an island in the Southern Ocean while he is in London. In the story *The Crystal Egg* (1897), the eponymous object turns out to be a window between parallel universes.

The problem of the orientation of our bodies in relation to journeys through an extra dimension was used by Wells in his tale *The Plattner Story* (1896-1897). Gottfried Plattner is an unfortunate science professor who, as the result of an explosion while he was in the laboratory trying to teach chemistry to his students, finds himself in the fourth dimension. When he returns from it he talks about what he saw during the days he spent outside of our space. The beings with no bodies, which he calls

"Watchers of the Living" spy on humanity, and the visions of our universe which he had while outside of it when nobody could see him are seen as the hallucinations of a madman, unhinged by the explosion. But the truly interesting part is that when he comes back to Earth, his body is back-to front, his orientation has changed, the parts from the left-hand side of his body, such as the heart, are now on the right and vice-versa. This proves that he has travelled to the fourth dimension and that his story is true. In the words of the story's narrator, "There is no way of taking a man and moving him about in space as ordinary people understand space, that will result in our changing his sides. Whatever you do, his right is still his right, his left his left... To put the thing in technical language, the curious inversion of Plattner's right and left sides is proof that he has moved out of our space into what is called the Fourth Dimension, and that he has returned again to our world". Yet another story of travelling to a fourth spatial dimension can be found in *The Stolen Nody* (1898).

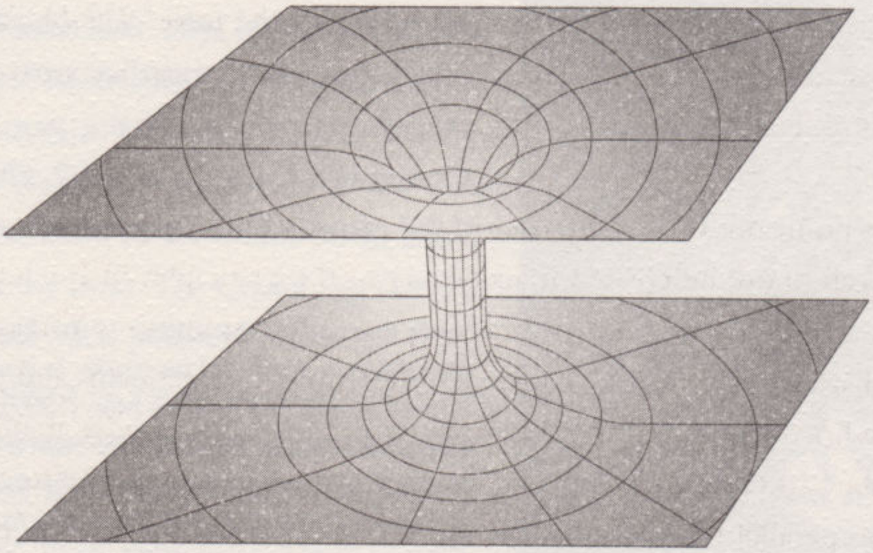
Wells's *The Time Machine* is for many the first work of fiction that includes a time machine, and one of the pioneering works in considering time as a dimension that can be travelled through. That concept also features in books such as *A Christmas Carol* (1843) by Charles Dickens, and *A Yankee in King Arthur's Court* (1889) by Mark Twain. However, it should be noted that Spanish playwright Enrique Gaspar (1842-1902), creator of some innovative theatre comedies, was to beat Wells to the literary creation of the time machine in his novel *El Anacronópete* (1887). In it, Sindulfo García, an inventor from Saragossa, builds a machine to travel through time – the anacronópete (from Greek *ana*, 'back'; *cronos*, 'time', and *petes*, 'one who flies'), with which he travels, together with other passengers, to periods of the past: The Paris Commune, Granada in 1492, China in the third century B.C., the last day of Pompeii

CARL LAEMMLE presents
H.G.WELLS
FANTASTIC SENSATION
THE INVISIBLE MAN
with
GLORIA STUART - CLAUDE RAINS
WM HARRIGAN - DUDLEY DIGGES - UNA O'CONNOR
HENRY TRAVERS - FORRESTER HARVEY
A UNIVERSAL PICTURE

Wells touched on the subject of the fourth dimension in several of his works. In one of his best-known novels, The Invisible Man *(1897), the main character acquires his invisibility from a "formula that is a geometric expression of the fourth dimension." The picture is the poster for one of the best cinematic versions of the story.*

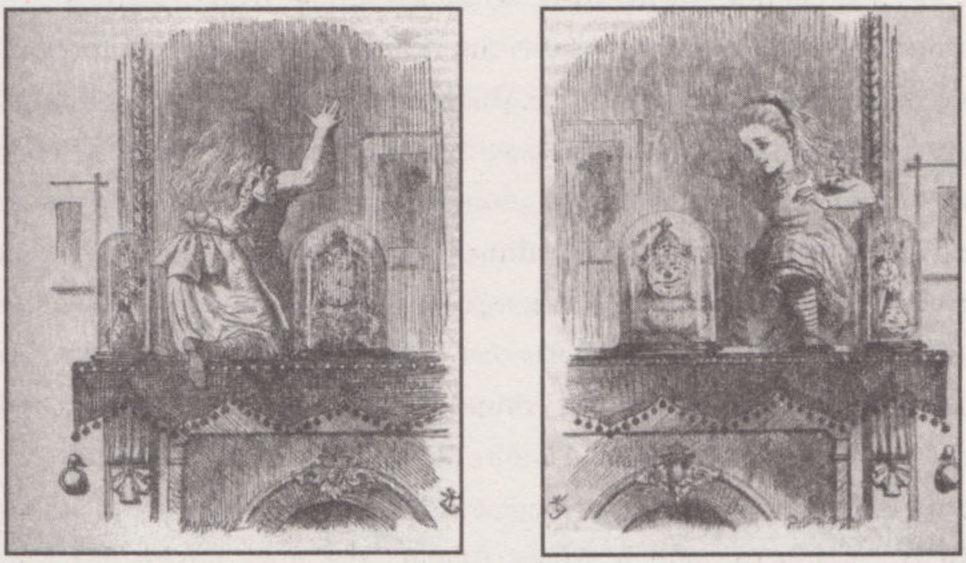
under Mount Vesuvius and the creation of the universe. Originally it was a zarzuela – a Spanish musical drama – the original manuscript of which dates from 1881 and is conserved in Spain's National Library.

When talking about mathematics in literature a mention of the author of *Alice in Wonderland* (1865) is almost inevitable. Lewis Carroll is the pseudonym under which English mathematician Charles L. Dodgson (1832-1898) wrote. And this book is not an exception. In his work *Sylvie and Bruno* (1889) there is a device, a watch, which allows time travel with certain limitations: when the time is changed on the watch it also changes in real life. But taking into account that Dodgson was aware of the work of Riemann and Lobachevski, and had even read the book *Transcendental Physics* by Zöllner due to his interest in spiritualism, it is not surprising that other references appear in his writing. In his book *Through the Looking Glass and What Alice Found There* (1871), the sequel to *Alice in Wonderland*, he uses Riemann's idea of "short-cuts", wormholes that connect two universes, ours and Wonderland in the case of this story, both connected through the mirror (or in the first instalment by the hole that Alice falls into). But, in this book Carroll also works with the concept of the change of orientation produced by travelling through a mirror. Before going through it, Alice is facing the mirror with her cat: "How would you like to live in looking-glass house, Kitty? I wonder if they'd give you milk in there? Perhaps looking-glass milk isn't good to drink..." Actually, it would not be since its molecules are either dextrogyral (they rotate the plane of polarised light to the right) or levogyral (to the left), and by passing though the mirror these



Wormholes joining two universes which, hypothetically, would allow us to travel between the two.

orientations would be flipped and become indigestible by stomach enzymes. Another more obvious and visual example of a change of orientation can be found when, Alice crosses to the other side of the mirror and sees that the books in the library are written backwards, like when you look at writing in a mirror. So, she sees the poem JABBERWOCKY written as: YꓘƆOWЯƎꓭꓭAႱ.



Pictures by illustrator John Tenniel of Alice Through the Looking Glass. *In the first image we can see how Alice, while looking in the mirror, has the clock to her right, but in the second, having already passed through to the other side, the clock is to her left as she looks in the mirror.*

It is interesting to see that as early as 1879 we can find literary references to non-Euclidean and multi-dimensional geometry in the novel *The Brothers Karamazov* by Russian writer Fiodor Dostoievsky (1821-1881). Iván Karamazov refers to them in his speculation on the existence of God.

> "I must point out one thing. If God does exist and really created the world, as we well know, he created it according to the principles of Euclidean geometry and made the human brain capable of grasping only three dimensions of space. Yet there have been and still are mathematicians and philosophers who doubt that the whole universe or, to put it more generally, all existence was created to fit Euclidean geometry. They even dare to conceive that two parallel lines that, according to Euclid, never do meet on earth do, in fact, meet somewhere in infinity.

In essence, I have a Euclidean mind, an Earthly mind, and therefore it is not for us to resolve things that are not of this world. And I advise you never to think about it, and most especially not about whether God exists or not. All such questions are completely unsuitable to a mind created with a concept of only three dimensions."

The idea of ghosts as beings that live in the fourth dimension and can enter and leave our universe as they like is used in the novel *The Canterville Ghost* (1887) by Irish writer, poet and playwright Oscar Wilde. In this comical and ironic vision of the period's ghost stories, it is written in reference to the ghost, that "there was evidently no time to be lost, so, hastily adopting the Fourth Dimension of Space as a means of escape, he vanished through the wainscoting, and the house became quite quiet." In other words, it did not go through the wall but passed into the fourth dimension.

The popularity of the fourth dimension also attracted the attention of British writer and poet Rudyard Kipling (1865-1936), author of *The Jungle Book* and winner of the Nobel Prize for Literature in 1907. He used the expression "fourth dimension" in at least two of his stories. In *An Error in the Fourth Dimension*, first published in *Cosmopolitan* magazine, there is actually no reference to it in the plot, and it simply appears in the title metaphorically. On the other hand, the story *The Brushwood Boy* (1895) tells of the main character's adventures in the world of dreams, which Kipling also denotes as the fourth dimension: "He hurried in desperation till he found himself utterly lost in the world's fourth dimension, with no hope of return."

But these examples are nothing more than a small sample of the use of the multidimensional themes in the literature from the golden age of the fourth dimension, which would last until the 1920s. To understand the magnitude of this influence, let's look at a few more cases.

In the novel *Lilith* (1895), by Scottish writer and poet George MacDonald, the protagonist, by means of the manipulation of mirrors, creates a dimensional door between our universe and another parallel one, which is inhabited by the spirits of the dead. American Ambrose Bierce, in his collection of *Mysterious Disappearances* (1893), tells of different cases of people, who are transferred from our space to another non-Euclidean one, where they get lost and fall into a kind of strange pocket in which they cannot see or be seen, hear or be heard or live or die. Russian Antón Chekov uses it as a term related to the paranormal in his tale *The Mystery*.

THE ORIENTATION OF THE TITANIC

When, in the film *Titanic* (1997), by James Cameron, we see the passengers boarding before it sets sail on its transatlantic journey, a lot of people can be seen waving goodbye unusually with their left hands, and there is even a person who is filming the historic event by moving the handle of his camera with the left hand, as if it was a left-handed camera –unheard of at the time. The reason for this strange situation is because the replica Titanic built for the film, faced the land on the starboard side, whereas ships board on the port side – as the word suggests. The scene was later inverted during editing, hence the strange image. In order to avoid errors, the director had to invert everything throughout filming. So, when the text appeared they had to change it for its mirror-image so that it read properly when inverted; for example, the name of the boat was written ƆIИATIT, so that it read TITANIC when inverted. All this made filming some scenes truly complicated. In fact, Cameron only built a detailed replica of half of the ship, the starboard side, which is why the scenes that took place on the port side were recorded on the starboard and later inverted during editing.

The British writers Ford Madox Ford and Joseph Conrad wrote *The Inheritors* (1901), about a race of beings from the fourth dimension who want to take over our world. In the story *The Hall Bedroom* (1905), by American Mary Wilkins Freeman, the main character passes into the fourth dimension while looking at a strange painting. In the second book of another famous American writer, F. Scott Fitzgerald, *Beautiful and*

Damned (1922), we can read: "It appeared to her that everything in the room was staggering in grotesque fourth-dimensional gyrations through intersecting planes of hazy blue." American poet William Carlos Williams, in his poem *Spring and All* (1923), writes: "What is the fourth dimension? It is the endlessness of knowledge; it is the imagination on which reality rides."

As can be seen, most of the authors mentioned are English-speaking authors, although they were not the only ones interested in these topics. There are three examples worth mentioning in this respect. The first is Frenchman Gastón de Pawlowski and his novel *Journey to the Land of the Fourth Dimension* (written between 1895 and 1912, with an extended edition in 1923). Following in the steps of Wells, Pawlowski uses fiction on the fourth dimension as a tool to discuss social issues. However, although this book is about a journey through time, Pawlowski does not consider it to be the fourth dimension, but rather he follows in the steps of Abbott and Hinton, understanding the city in the fourth dimension in the sense of a space. The

On both sides of the Atlantic, the fourth dimension became a constant in literature at the beginning of the 20th century. Frenchman Marcel Proust (above left) covered it in his masterpiece In Search of Lost Time, *while American F. Scott Fitzgerald (above) used the theme in* The Beautiful and Damned. *The Russian Antón Chekov (left) linked the fourth dimension to paranormal phenomena in* The Mystery.

French writer was also deeply influenced by Zöllner's spiritualism and by his work-*Transcendental Physics*, which was much acclaimed among French intellectuals at the beginning of the 20th century, especially among cubist painters.

With another Frenchman, Marcel Proust (1871-1922), we find the dual handling of the fourth dimension as space and time in his work *In Search of Lost Time* (1913), producing what we could call the materialisation of time, which is no more than what we have described as static space-time. It describes a church which is actually no more than a kind of time machine: "[It was] an edifice occupying a space with, so to speak, four dimensions – the fourth being Time – extending over the centuries its nave which, from bay to bay, from chapel to chapel, seemed to vanquish and penetrate not only a few yards but epoch after epoch from which it emerged victorious, hiding the rugged barbarities of the eleventh century in the thickness of its walls."

Finally, a devotee to the subject of the fourth dimension was Mexican poet Amado Nervo (1870-1919). He wrote an article on the fourth dimension in which we can read: "Our conscience is not, like our senses, built according to the vision of the three-dimensional world, quite the contrary, it leads us to that 'fourth dimension', which, in summary, is no more than the complement necessary for a total understanding of the universe... The poet, who is a supreme artist...spends hours at a time in a world of four dimensions. Poetic ecstasy, like all ecstasies, is no more than access to a new dimension... Man's 'soul' would be no more than our extension in an unknown dimension."

After the theory of relativity

Albert Einstein published his theory of general relativity in 1915, and in the 1920s it was already becoming so popular that it diverted attention from the fourth dimension, as it had been understood previously, to relative space-time. The theory of relativity recuperated a large part of the interest in the fourth dimension while adding new elements, both in the understanding of our universe and aspects which are fascinating for the interested public, such as the twins paradox and other similar subjects. Such paradoxes are the consequence of the fact that in relativity, time and space are interrelated, and time is not another dimension, as is understood within a time-space continuum. The theory of relativity quickly gained a large following among intellectuals and artists.

Bit by bit, the interest in a spatial fourth dimension was waning in favour of the

THE TWINS PARADOX

Without getting into technical discussions on the theory of relativity, the twins paradox states the following: There are two twin brothers; the first, an astronaut, undertakes a long journey to a distant star in a spaceship at speeds close to the speed of light, while the other remains on Earth. When the astronaut twin comes back, it turns out he is much younger than his brother. According to the theory of relativity, this is because time slows down with speed (personal expansion); consequently, while the astronaut was travelling at high speeds in his spaceship, time passed more slowly for him than for his brother, and therefore, he aged more slowly. The film *The Planet of the Apes* (1968), based on the novel of the same name by Pierre Boulle, used the twin paradox in its plot. At the end of the film it is discovered that the astronauts had actually returned to Earth, but a long time after they left, so much so that mankind had lost its supremacy, and the apes were now the planet's dominant species. However, for the astronauts only a few years had passed.

relativity version, becoming anecdotal between 1950 and 1970. In any case, writers captivated by the ideas of Hinton and Abbott continued to appear regularly, but, above all, it was the science fiction generation, which was in full development by this time, where all existing concepts were being used as literary devices, among them multidimensional space in any of its growing number of definitions. Following those decades, the passion for this concept returned in mathematics.

Despite the popularity of the theory of relativity, it had a double handicap. Firstly, it is a very complicated theory, even for the scientists themselves, and secondly, the theory itself creates an insurmountable obstacle to travelling back in time, a very popular subject among writers, who instead continued to use the idea of the (non-relativity) time-space continuum, whatever they chose to call it.

American writer William Faulkner (1897-1962), like many other modernists before him, used the concept of the fourth dimension in his work. The novel *As I Lay Dying* (1930) employs a space-time structure, translating the past into the present, and uses the fourth dimension as a metaphor for memory: "He looks up at the gaunt face [of his mother] framed by the window in the twilight. It is a composite picture of all time since he was a child." This almost spatial understanding is something that can be found right at the beginning of the book: "The path runs straight as a plumb-line, worn smooth by feet and baked brick-hard by July, between the green rows of cotton, to the cottonhouse in the center of the field, where it turns and circles the cottonhouse at four soft right angles and goes on across the field again, worn so by feet in fading precision."

One book that is very individual in its use of static time-space is the wonderful *Slaughterhouse Five* (1969), by American Kurt Vonnegut (1922-2007), in which the author discusses, through fiction, his experience in the Second World War as a soldier and prisoner of war who survived the firebombing of Dresden by the Allies when the war was practically over. This event deeply affected the author, who, through the main character, discusses this vengeful massacre. The protagonist is "spastic in time", travelling forwards and backwards in time, reliving the events of those days. The journey through the fourth dimension comes to an end thanks to the Tralfamadorians, a strange extraterrestrial race for whom time is just another dimension in which to move.

British humorist writer P.G. Wodehouse (1881-1975), in his story *The Amazing Hat Mystery* (1936) uses the fourth dimension as a tool for humour. It is about two English gentlemen who both buy hats and the mystery that surrounds them. At the beginning the narration is as follows: "You already know how things are: if something strange happens to someone, he asks someone who has their head screwed on, who shakes their head and says: 'Ah! The fourth dimension.'"

William Faulkner, one of the most revered authors of American literature. There are frequent jumps through time in his work.

Russian-born American Vladimir Nabokov (1899-1977) also plays with the change of orientation produced in the dimensional twists of the novel *Look at the Harlequins!* (1974).

Swedish writer Lars Gustafsson (c. 1936), awarded with the German Goethe literary prize in 2009, writes in his most acclaimed work, *The Death of a Beekeeper* (1978): "Paradise? I experienced all that recently. Paradise must consist of the stopping of pain. That means, however, that we live in Paradise as long as we have no pain! And we don't even know it. Happy and unhappy people live in the same world, and they don't even know it! I have the feeling as if during the past months I have been walking around my own life in a fantastic, mysterious maze, and now I have returned precisely to that spot at which I began. But, since I moved outside the normal dimensions, right and left somehow got exchanged. My right hand is now my left one, my left hand my right one. Returned into the same world and see it now as a happy one."

Borges and the fourth dimension

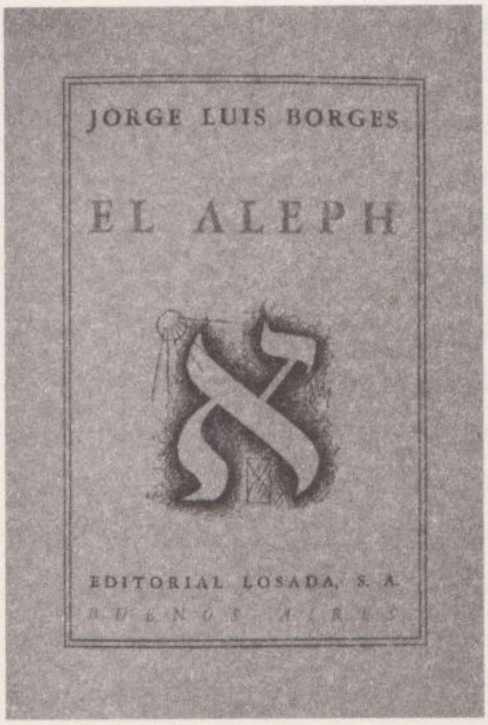
One essential reference must be made to Argentinian writer Jorge Luis Borges (1899-1986) whose profound interest in mathematics was extensively reflected in his work. Subjects such as infinity, rational numbers, paradoxes, labyrinths, tessellations of the plane, combinatorics and set theory are a few of the mathematical issues that appear in it. The fourth dimension was also one of the subjects, which attracted his attention, and he even wrote an article for several literary magazines titled "The Fourth Dimension", in which he sought to explain it from a geometric point of view. In it he comments on *New Era of Thought*, a book by Hinton, and *ABC of the Fourth Dimension*, by theosophist and architect Claude F. Bragdon.

There are other works by Borge, which contain explicit references to the fourth dimension. In the story *Tlön, Uqbar, Orbis Tertius*, from his book *Fictions* (1944), it is written that the letter that explains the mystery of Tlön appears in "Hinton's book". Regarding Tlön's geometry it says that "it knows not of parallels, and it postulates that a man stretching changes the forms surrounding him". Rodolfo Mata, in his article "Borges and the adventure of the fourth dimension", also says that "the surprising language of Tlön, in which the replacement of nouns with verbs is given for one hemisphere - moon becomes to moon - while in the other hemisphere nouns are substituted by adjectives - *moon* becomes light-air on dark-round - may have its roots in *Tertium Organum* (1911) by P.D. Ouspensky. According to this Russian mystic and philosopher, in order to adequately understand temporary relationships established by the fourth dimension, a language without verbs would be necessary: "If life in the third dimension equals movement in the fourth, movement in the third disappears in the fourth".

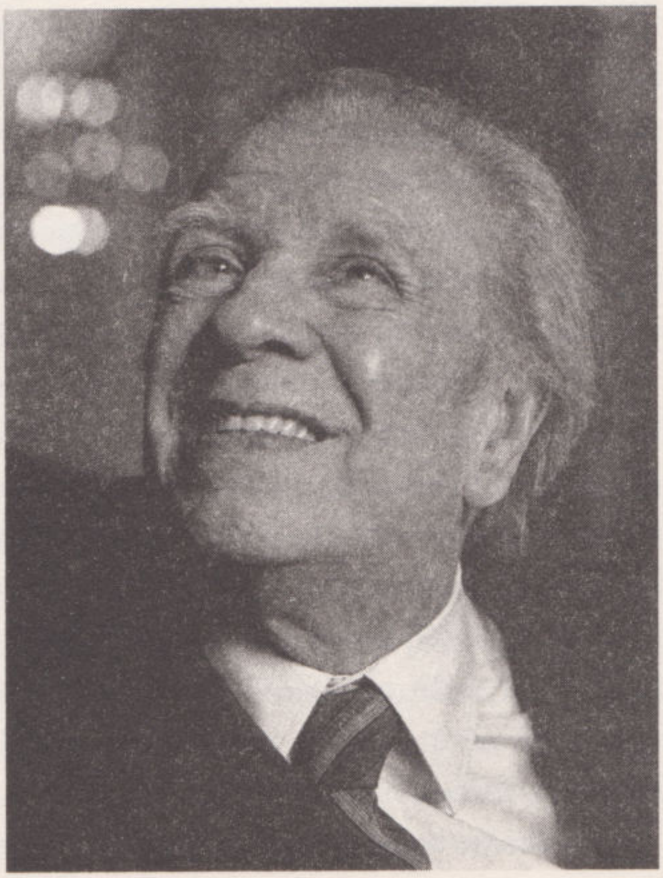
Borges quotes Ouspensky again in *Time and J.W. Dunne,* included in *Other Inquisitions* (1952). In his explanation of Dunne's theory of infinite dimensions of time *Tertium Organum,* the Russian writer writes that "the future already exists", in other words, he is establishing the static notion of time-space.

In the fantasy tale *There are More Things*, from *The Book of Sand* (1975), the main character's uncle lends him "the Hinton documents, which seek to demonstrate the reality of a fourth dimension in space, which the reader can sense by means of complicated exercises with coloured cubes" and on other occasions quotes Hinton's cubes to give the story a feeling of dimensional fantasy. The main character investigates the house in which his uncle died, which turns out to be a door to other dimensions. The story of *The Book of Sand* itself starts with the words: "The line consists of an infinite number of points; the plane, of an infinite number of lines; the volume, of an infinite number of planes; the hypervolume, of an infinite number of volumes…".

In *Ibn Hakkan al-Bokhari, Dead in his Labyrinth,* from *The Aleph* (1949), he writes:



The cover of an edition of The Aleph, *a story in which Jorge Luis Borges suggests a new vision of the universe.*



"'I chose to forget your absurdities and think about something sensible. The theory of mathematical sets, let's say, or the fourth dimension of space,' observed Dunraven."

And, let's finish with another of Borges' poems, *Adrogué*, which comes from *The Maker* (1960), and in one of its verses it says: "Beyond random and death / they last, and each one has its story, / but all this happens by this luck / of fourth dimension, which is the memory".

Science fiction

There is no doubt that it is in science fiction that the subject of the fourth dimension was used most during the 20th century. Among the authors that have made use of it we can find the greats of fantasy literature: I. Asimov, G. Bear, N. Bond, A.C. Clarke, A. Deutsch, G. Gamov, F. Lenz, H.P. Lovecraft, L. Padgett, F. Pohl, R. Rucker, C. Simak and M. Smith, among many others.

It is worth mentioning two works strongly linked to the fourth dimension: The short story *And He Built a Crooked House* (1940), by Robert A. Heinlein, in which an architect builds a house that is a demonstration of the hypercube in the third dimension (which is discussed in the following chapter), and which, once built, is warped in the fourth dimension, leaving the architect inside. The other is a children's story by Madeleine L'Engle, *A Wrinkle in Time* (1962), in which the hypercube is also described.

Chapter 7

Visualising the fourth dimension

> *In the same way that we can make on a plane the perspective of a figure having three dimensions, we can make that of a four-dimensional figure on a surface with three (or two) dimensions. We can even take from a single figure several perspectives from several different viewpoints…* [and studying the "whole" of these perspectives] *we may say that we can represent the fourth dimension.*
>
> *Henri Poincaré,* Science and Hypothesis *(1902)*

There are people who think, due to the apparent three-dimensionality of our brains (we say apparent since it could be that is no more than the cross-section of a four-dimensional brain in our world), that it is impossible for us to visualise the fourth-dimension. Certainly, it is a complex problem, but we could think, as Poincaré did, that in the same way that painters use two-dimensional canvasses to paint three-dimensional shapes and scenes, or engineers use plane projections to study the geometry of the tools, machinery and buildings they are working with, we could try to visualise 4D objects by 'painting them' in 3D using projections or other techniques. Even studying the '3D paintings' embodied by different points of view, it is possible to understand, or visualise in some sense, the four-dimensional object.

At the end of the 19th century and the beginning of the 20th century, one of the main problems of multidimensional spaces was the visualisation of those spaces and their objects. And the representation of the hypercube, the four-dimensional version of the cube, became a symbol of that visualisation. In that time, a huge effort was made to study the hypercube and, in general, *n*-dimensional polytopes by people such as Charles H. Hinton, Claude Bragdon, W.I. Stringham, Alicia Boole Stott (Hinton's sister-in-law), American Henry P. Manning (author of the book *The Fourth Dimension Simply Explained*), Frenchman Esprit Jouffret (author of several papers on the fourth dimension) and Henri Poincaré, among many others.

Essentially, the techniques for visualising the fourth dimension consist of stepping down to the third-dimension by means of different types of projections, the creation of sections, or using the deployment method. The techniques which we are going to explain, as well as the images generated by them, were already well-known and commonly used in the period under discussion. The descriptions of the different methods of visualisation that we are going to carry out are quite intuitive, and, as with the rest of the book, we will make use of dimensional analogies as a fundamental tool for understanding it.

The hypercube and the hypersphere

The hypercube, also known as the tesseract (a term coined by Charles H. Hinton in his book *A New Era of Thought*), is a generalisation of the cube in the fourth dimension.

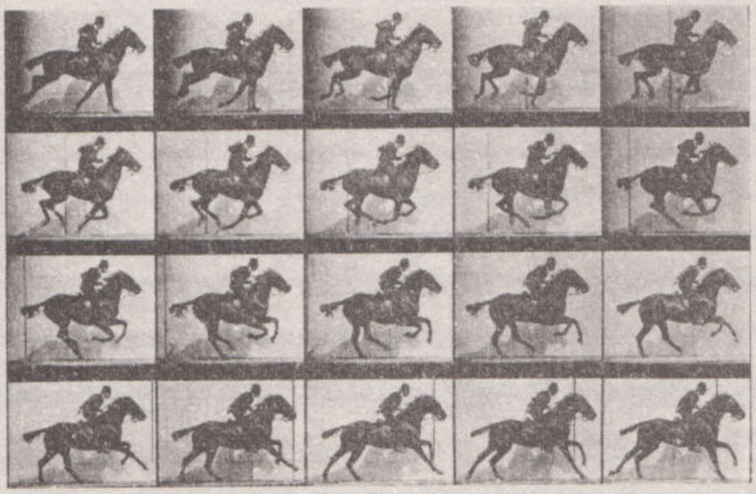
Returning to the ideas of Chapter 1, let's say that in the zero dimension we have a point, in particular, it would be our '0-cube' or cube in the 0-dimension. If the point is now on a straight line (1-dimensional space) and moves a certain distance on it, we will get a segment (which will be a 1-cube). If we are on the coordinate

DO 3D PROJECTIONS HELP TO VISUALISE THE FOURTH DIMENSION?

When explaining the visualisation of four-dimensional objects in three dimensions, or even two, there are those who think it impossible to achieve a real understanding of the object in question in this way. While recognising that this idea is, to an extent, true, it is no less so that as humans we are accustomed to representing our world in two dimensions by means of paintings, photography and, if we have time, cinema. However, we do not tend to question these flat representations of reality. It is even possible to obtain information on reality through these 2D images, particularly if we include the variation of perspective and time. Let's have a look at a couple of simple examples to illustrate this point. The first is shadow plays, which, despite the simplicity of the flat, black images, does not cause us any problems in recognising the shapes and understanding the story being told.

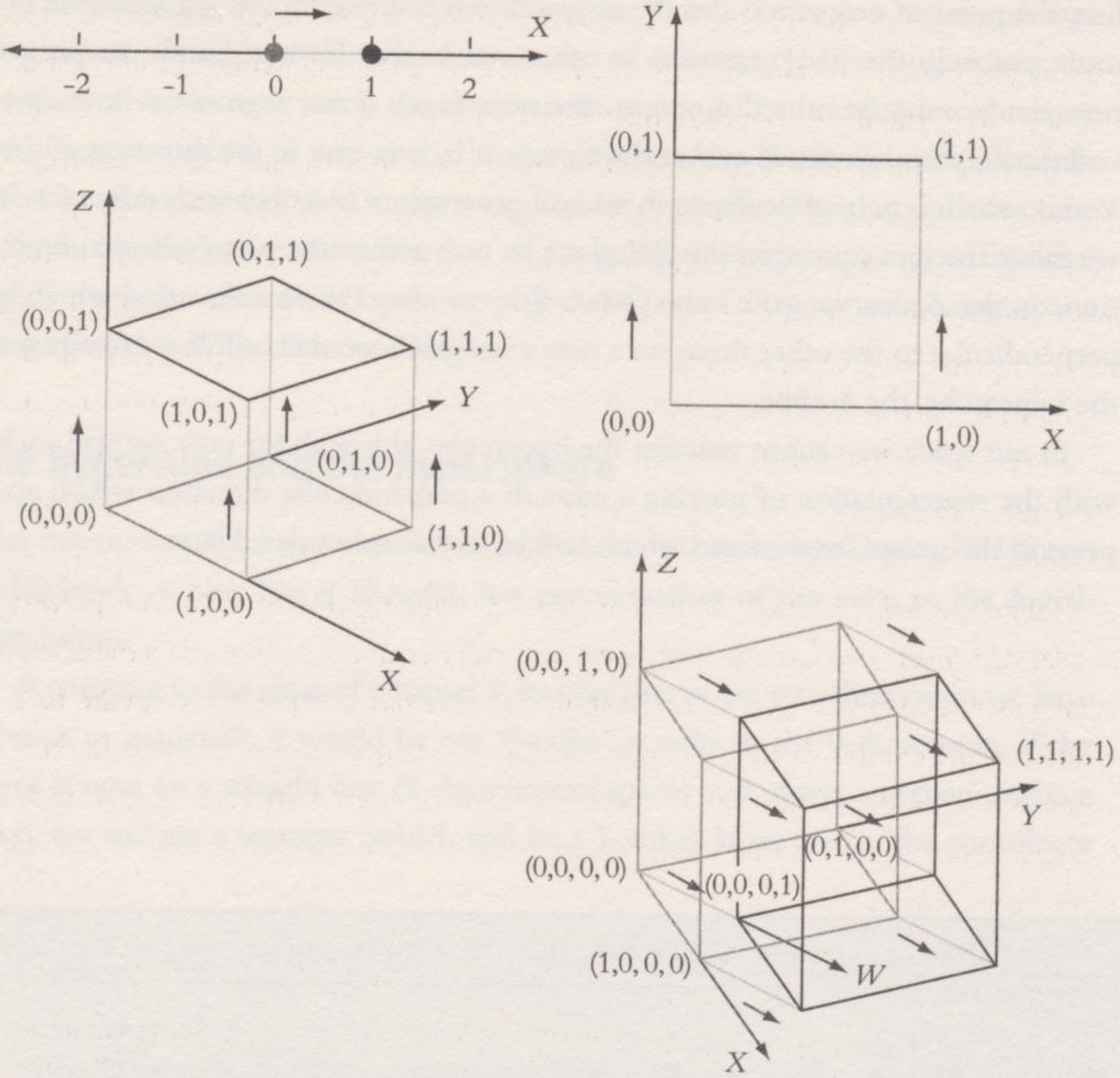
The second is the famous dispute on the movement of the horse. Up until 1870 Californian horse racing fans were locked in a debate on whether or not there is a moment in a horse's gallop when none of the hooves are touching the ground. The dispute was resolved by a series of images line, the point of origin is 0 and the displacement is one unit, our segment will be more precisely the [0,1] segment, in other words, that formed by all the points between 0 and 1 (see the diagram on the next page). If our segment is now on a coordinate plane, on the X axis, and we move it by one unit in the direction of the Y axis, which is perpendicular to it, we will get a square (2-cube) with sides of 1. If we move the unit square on the XY plane by one unit in the perpendicular direction, on the Z axis, we get a cube (3-cube); by moving this in a direction which is perpendicular to the other three, on a new axis which we will call W, we finally get the hypercube, the 4-cube.

In our space we cannot visualise the hypercube, although for now we will stick with the representation of moving a cube in a perpendicular direction which appears in the image overleaf and which will be explained in detail later.



Series of stills taken by Muybridge showing the motion of a horse. In them it can be seen that at one point in the horse's motion the animal is moving forward without touching the ground.

taken by British photographer Eadweard Muybridge (1830-1904), in which it was demonstrated that there was a moment in a horse's running motion when none of its hooves touch the ground.



The segment (line), the square, the cube and the hypercube, with sides of 1, in the corresponding coordinate space.

It is intuitively clear that each *n*-cube, cube with *n* dimensions, is obtained by moving the cube from one dimension less, the (*n*-1)-cube, in a direction perpendicular to the previous ones. However, when we work mathematically in *n*-dimensional coordinate space, the unit *n*-cube can be expressed as points with coordinates with values between 0 and 1, in other words,

$$n-cube=\left\{\left(x_1,\ldots,x_n\right)\in\mathbb{R}^n : 0\leq x_1,\ldots,x_n\leq 1\right\}.$$

Each *n*-cube is formed by a family of elements from inferior dimensions, *k*-cubes, for $0\leq k\leq n$. For example, the hypercube is formed by the following elements: points (the vertices, or corners), segments (the edges), squares (square faces), cubes (cubic faces) and the hypercube itself. In order to try to understand its nature we can start by analysing the elements that make it up, using the following reasoning as an analogy (using the image opposite may help).

First of all, the elements of a 1-dimensional cube is a segment, or line. A segment, has two vertices and, of course, the segment itself. Now, by moving it in a perpendicular direction to generate a square, we have the two initial vertices and the two final ones; consequently, in general the vertices are multiplied by two when moving it. So, a square has 4 vertices, a cube 8, and a hypercube 16. But, also, if we count the edges of the square, we have the initial segment, the final one, plus those formed by each vertex in when it is moved, so the square has 1 + 1 + 2 = 4 edges. Likewise, the cube will have 4 + 4 + 4 = 12 edges, and the hypercube, 12 + 12 + 8 = 32 sides.

Below we will calculate the square faces. By moving the square in the perpendicular direction, we have the initial and final squares, and with the movement each edge generates a new square face, therefore the cube has 1 + 1 + 4 = 6 square faces, while the hypercube would have 6 + 6 + 12 = 24 square faces. Finally, the cube, when it is moved, generates the initial and final cubes in the hypercube, and each face generates another cube in the movement; so, in total, 1 + 1 + 6 = 8 cubic faces. In summary,

Elements / Dimension	Vertices	Sides	Square faces	Cubic faces	Hypercubes	Total number of elements
0	1	0	0	0	0	1
1	2	1	0	0	0	3
2	4	4	1	0	0	9
3	8	12	6	1	0	27
4	16	32	24	8	1	81

The hypersphere is the equivalent of the sphere in the fourth dimension. But, to define it, first we have to understand what a sphere is. It is formed by all the points at the same distance (radius) from a given point (centre). Using analytic geometry, in other words, with coordinates $O = (0,0,0)$ with radius r, it would be expressed mathematically as

$$S^2 = \left\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = r^2\right\}.$$

Also, we can see that the sphere is a 2-dimensional surface.

FORMULAE FOR DETERMINING THE ELEMENTS OF AN *N*-CUBE

With a little more mathematics we can obtain general formulae for determining the elements of an n-cube. Let's use $E(k,n)$ to express the number of k-cubes in an n-cube. To calculate $E(k,n)$ first we will look at the first number of k-cubes touching a given vertex. As n sides extend from each vertex, and to define a k-cube that touches that vertex, we simply need to take k of those sides, so the number of k-cubes that touch the vertex will be equal to the number of possible ways to take k within the group of n sides that come out of the vertex. This is a combinatorial number,

$$\binom{n}{k} = \frac{n!}{k!(n-k)!},$$

where $n!$ is the factorial of n, in other words the product of $n! = n\,(n-1)\,(n-2)\ldots 3 \cdot 2 \cdot 1$. As we have 2^n vertices, in principle they will be

$$2^n \binom{n}{k}$$

k-cubes, but each k-cube contains 2^k vertices, which means that we are counting each k-cube exactly 2^k times, and we will have to divide by that amount. Ultimately,

$$E(k,n) = 2^{n-k} \binom{n}{k}.$$

And, in general, the number of elements of an n-cube will be:

$$E(0,n) + E(1,n) + L + E(n-1,n) + E(n,n) =$$

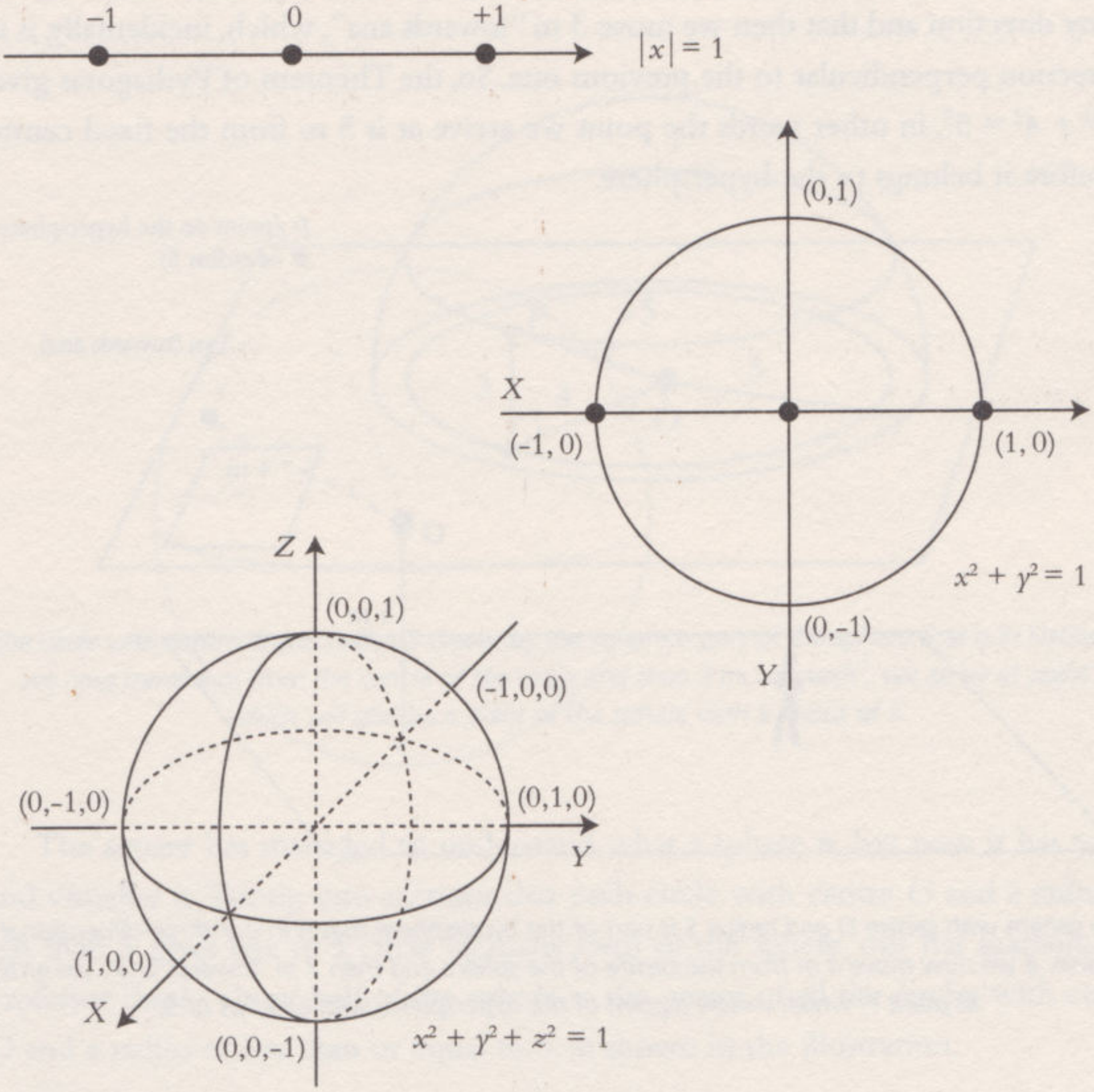
$$= 2^n + 2^{n-1}\binom{n}{1} + L + 2^{n-k}\binom{n}{k} + L + 2\binom{n}{n-1} + \binom{n}{n} = (2+1)^n = 3^n.$$

The reader can check that the above box fits the formulas obtained.

In general, for any $(n+1)$-dimensional space, the corresponding n-sphere is formed by the points of the $(n+1)$-dimensional space, which are the same distance from its centre. In mathematical terms,

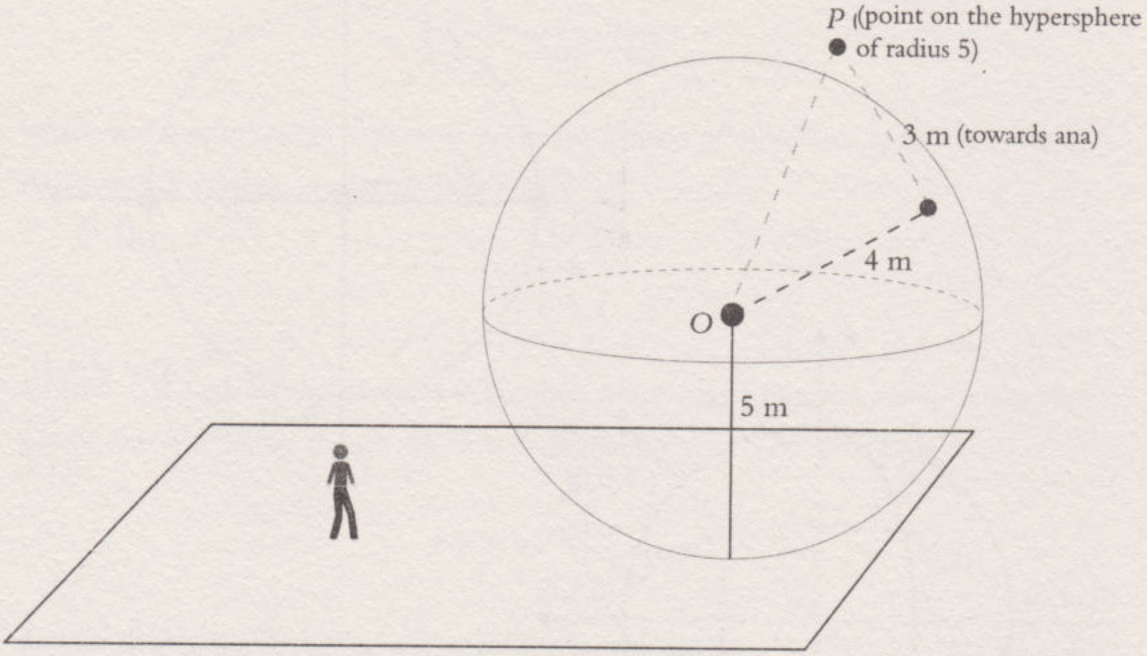
$$S^n = \left\{ \left(x_1, \ldots, x_{n+1}\right) \in \mathbb{R}^{n+1} : x_1^2 + \cdots + x_{n+1}^2 = r^2 \right\}.$$

If we are in dimension one, the straight line, then the 0-sphere with the centre of origin 0 and radius of 1, are the points $\{-1,1\}$, as shown in the illustration. On a plane, the 1-sphere is no more than the circumference of the centre of origin and a radius of 1, while in three-dimensional space the 2-sphere will be that which we normally understand by a sphere.



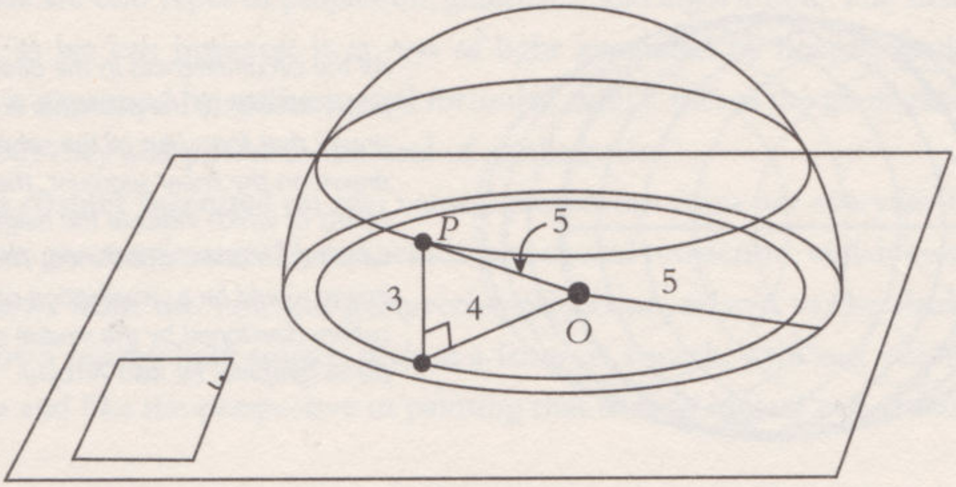
The n*-spheres with a radius of 1 centred at the origin in the coordinate space of dimension* (n+*1), for* n *= 0,1,2. Where* n *= 0, the sphere has a dimension of 1; where* n *= 1, dimension 2; where* n *= 2, dimension 3.*

This is where we come across the problem of how to visualise and also try to better understand what the hypersphere is. Let's suppose that the spatial fourth dimension exists, we are on a football pitch looking at a 5-m high mast and and we want to imagine what a hypersphere with a centre at that point (the tip of the mast) and has a radius of 5 would be like. What we already know and can even visualise is a normal sphere (a 2-sphere) centred at that point and with a radius of 5 (as shown in the image below), which are the points in our three-dimensional space that are 5 m from the centre, then it is clear that they also belong to the hypersphere. But how can we visualise or locate the rest of the points of the hypersphere, which are not in our space? Now let's assume that we move 4 m from the centre of the sphere, in any direction and that then we move 3 m "towards ana", which, incidentally, is in a direction perpendicular to the previous one. So, the Theorem of Pythagoras gives us $3^2 + 4^2 = 5^2$, in other words the point we arrive at is 5 m from the fixed centre, therefore it belongs to the hypersphere.



The sphere with centre O and radius 5 is part of the hypersphere that is in our three-dimensional universe. If we now move 4 m from the centre of the sphere and then 3 m 'towards ana', we arrive at point P, which will be a point of the hypersphere with a radius of 5.

This is how we obtain all the points of the hypersphere. In order to understand a little better the idea that we have introduced, we are going to carry out the same process on a experiment surface, Flatland. Let's suppose that the square that is the protagonist of Abbott's book would like to visualise a sphere with a centre at point O of the plane and a radius of 5. Before starting, it paints, in its flat universe, a circle with a radius of 5, which it knows belongs to the three-dimensional sphere; in fact, it is the part of it which is found in Flatland. Now it acts in the same way as us; it moves in any direction from the centre, for example, a distance of 4 m, and then imagines moving 3 m 'upwards'. Then, using the Theorem of Pythagoras (which it understands luckily) it knows that the point it has arrived at is a point of the sphere (see the illustration below). Moreover, with a circle of radius of less than 5, for example 4, from those points on that circle the square obtains another circle in the upper part of the sphere (in fact, it is a flat cross-section of it), located 3 m above Flatland. Furthermore, it would find another smaller circle if it went 'downwards'.

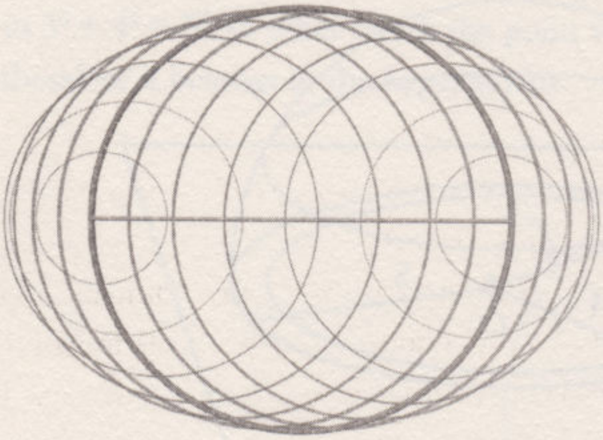


The circle with centre O and radius 5 drawn by the square is part of the sphere that is in Flatland. If we now move 4 m from the centre of the circle and then 3 m 'upwards', we arrive at point P, which will also be a point of the sphere with a radius of 5.

The square has managed to understand what a sphere is, but now it has to try and visualise it. Taking into account that each circle with centre O and a radius of less than 5 gives rise to a circle (actually two) of the sphere, our flat mathematics professor thinks about half of the sphere as the group of all the circles with centre O and a radius of less than or equal to 5, as shown in the illustration.

The semi-sphere imagined on the plane as all the circles of the plane with a radius less than that of the sphere (graphics by Josu Arroyo).

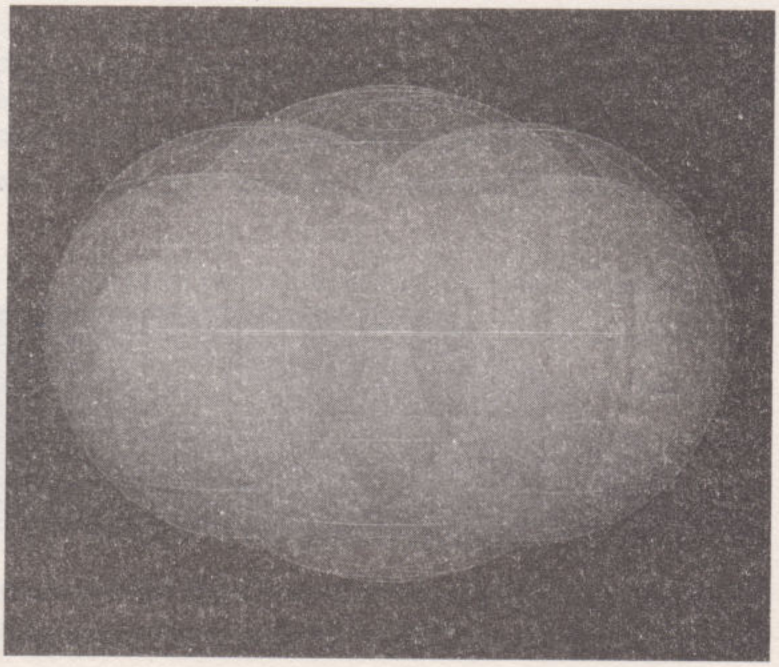
The square can visualise that image well in its mind, but it finds it difficult to represent, so it decides to go a little further and separate all those circles along the length of a segment (the segment with ends -5 and 5 in a straight line), so that each point on that segment tells us the height *h* from the plane, 'upwards' in positive and 'downwards' in negative, and at that point the sphere's circle, which is at height *h* (whose radius *c* is the positive number verified by the Theorem of Pythagoras: $h^2 + c^2 = 5^2$). The following image is obtained in the same way.



All the circumferences in the direction perpendicular to the plane (up and down) that form part of the sphere, drawn on the linear segment, the points of which indicate the height at which each one is positioned. This image would be a visualisation of the sphere developed by the square on the plane (graphics by Josu Arroyo).

Going back to the case of the hypersphere with a radius of 5 in the fourth dimension, and continuing in a manner similar to the Flatlander, we can think of the semi-hypersphere as the family of all the spheres with centres that are the tip of the mast and with radii less than or equal to 5. Ee can represent the hypersphere as all the spheres found at various heights *h* in the direction of ana or kata.

All the spheres in the direction perpendicular to three dimensional space (towards ana and towards kata) which form part of the hypersphere, drawn on the linear segment, the points of which indicate the height of each sphere. This image is a representation of the hypersphere in our three-dimensional space (graphics by Josu Arroyo).



Orthogonal projection

One of the methods used to represent a four-dimensional object, in this case the hypercube, in 3D, or even in 2D, are mathematical projections that transform four-dimensional space into three-dimensional space on a plane. Generally, we can use mathematical projections to transform any *n*-dimensional space into another of fewer dimensions.

There are two types of projection, geometric and algorithmic. The first is more natural, as we can interpret it as rays of light generated by images, shadows; the second is expressed by mathematical formulae, which means the geometrical sense is lost, but they add power to the tool of mathematics.

This chapter is centred on two natural projections from our everyday lives and, therefore, geometric ones. These include orthogonal projection, which corresponds to shadows from the sun, and perspective projection, related to the shadow produced by a nearby light source, such as a lamp or a torch, with our vision in perspective and like the perspective in painting that tries to imitate our vision.

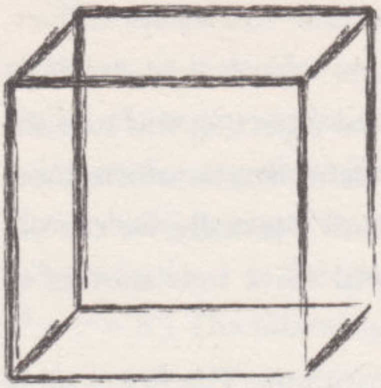
ALGORITHM VERSUS LOGARITHM

It is not unusual to find news stories in which the terms 'algorithm' and 'logarithm' are confused. For example, in a 2009 report we could read: "Google will identify unhappy employees with a logarithm". Where there is doubt it is always a good idea to consult a good dictionary. Doing so, we would find that a logarithm is the "exponent by which a fixed number, the base, has to be raised to produce that number". In other words, the logarithm of a number x, with respect to a given base b, is the exponent n that needs to be raised b to give that number x. In mathematical terms,

$$log_b x = n \Leftrightarrow x = b^n.$$

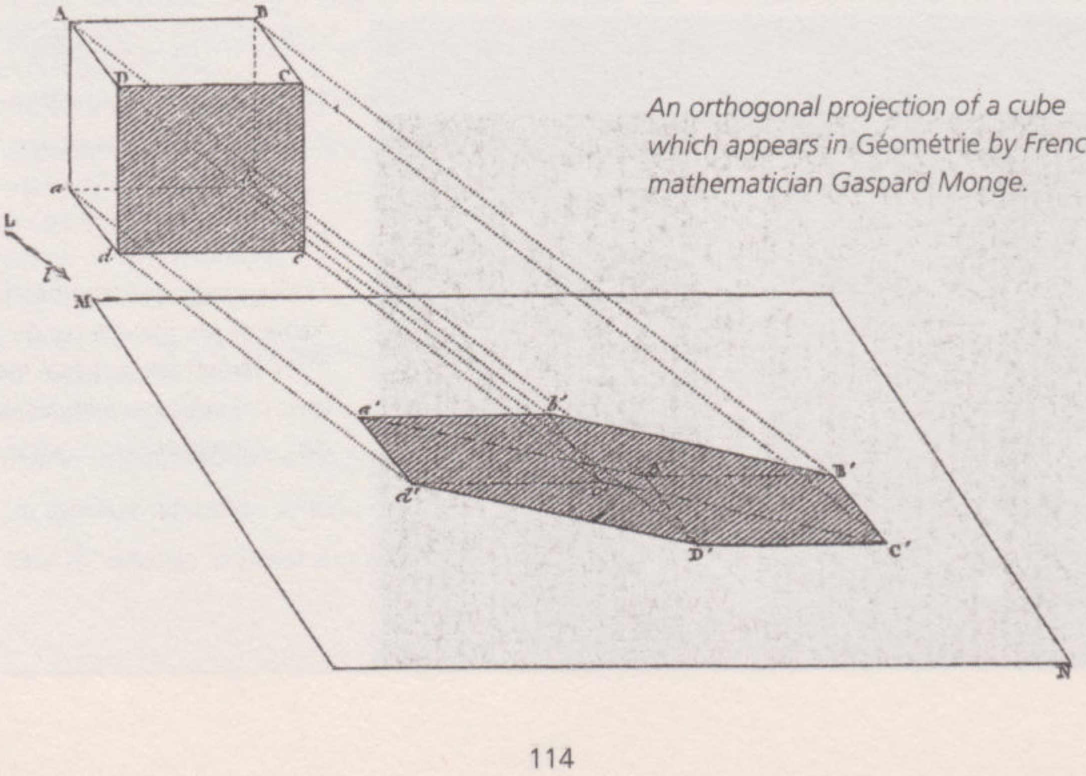
For example, as $3^2 = 9$, the base 3 logarithm of 9 is 2, and is written $\log_3 9 = 2$.

An algorithm, on the other hand, is an "an ordered and finite group of operations, which allows the solution to a problem to be found", be it in mathematics, computer sciences or other related disciplines, but "method and notation in the various forms of calculus" is also called algorithm.



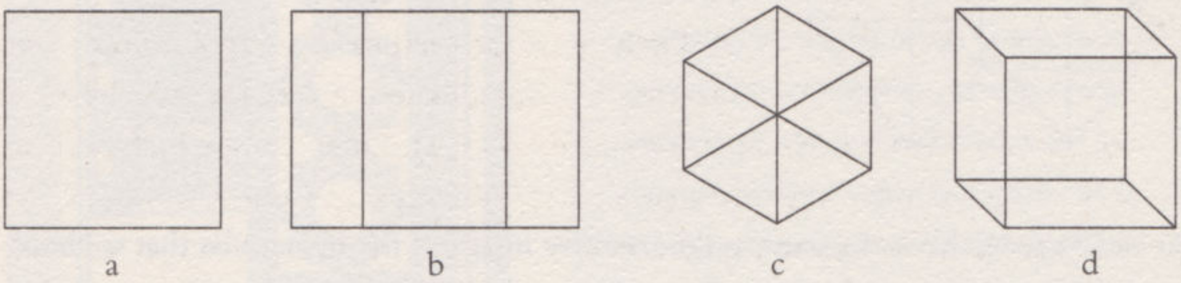
Before starting with orthogonal projection it is worth asking ourselves what we are drawing, and have done since our childhoods, when we want to represent a cube. It is normally something similar to the cube in the diagram, but are we really aware of what we are drawing? The sketch to the left is the orthogonal image of a cube.

Orthogonal projection is an application of n-dimensional coordinated space, for whichever n dimension we want, in one of its subspaces of $(n-1)$ dimensions, which consists of projecting onto this according to a certain direction. In other words, all the points that are on the same straight line in a given direction are projected onto the point in the inferior $(n-1)$-dimensional space, which intersects that straight line. If we consider three-dimensional space, the subspace onto which we project is a plane. We can think of the image of an object obtained using orthogonal projection as the shadow of the object produced by a source of parallel rays of light, in the direction of the projection, onto the projection plane (see the illustration below). For example, as the Sun is a very long way from Earth, its rays can be considered to be parallel and they fall on the Earth in a particular direction. Therefore, the shadows produced by the objects are orthogonal projections of them. Of, course, if we change the direction of projection we get different flat images of the same object.



An orthogonal projection of a cube which appears in Géométrie *by French mathematician Gaspard Monge.*

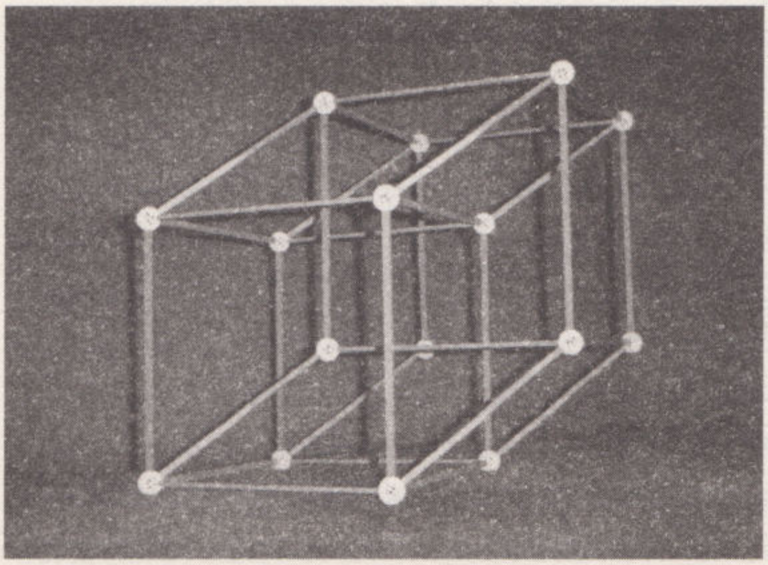
Now let's consider a three-dimensional cube and project it onto a plane. To better visualise the projection, let's imagine that we have a cubic frame, in other words, rods that represent the lines making up the cube and which show its structure. Projecting from different directions we get the following shapes. As can be seen, they fit very well with the intuitive images we have used throughout the book: A square which moves in a perpendicular direction generates a cube.



Orthogonal projections of a cube following a direction: a) perpendicular to two faces of the cube and parallel to the other four; b) parallel to only the upper and lower faces of the cube; c) parallel to the diagonal; d) not parallel to the faces or the diagonal.

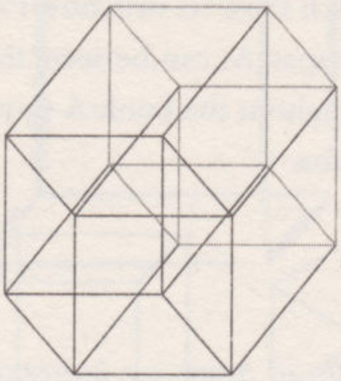
In this particular case the properties of the orthogonal projections can be appreciated perfectly: they translate straight line segments into straight segments, or points, and preserve parallels. Also, parallel segments of an equal length are projected into parallel segments of an equal length.

If we now orthogonally project a four-dimensional hypercube (or rather its frame), into three-dimensional space, we get the three-dimensional shape that appears in this image.

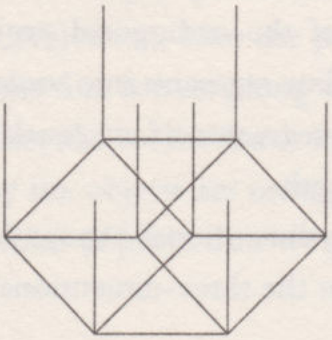


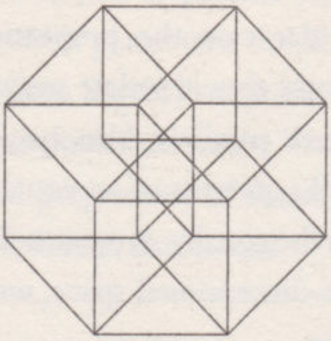
Orthogonal projection of the frame of the hypercube in three-dimensional space, made with the educational building tool, Zometool.

If we orthogonally project it again onto a plane we get the classic image of the hypercube.

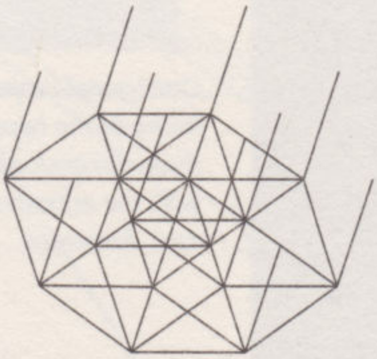


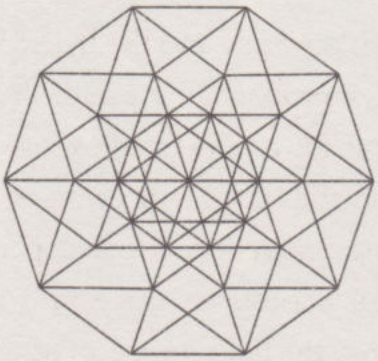
As can be seen, this is the same as the intuitive image of the hypercube that we have used previously when thinking about this as a cube that has moved in a perpendicular direction. Let's go back to this idea. If a cube is moved in a perpendicular direction it generates a hypercube, a flat image of this would be the following:





Based on the direction we use to represent this movement, the image of the hypercube will vary in beauty, depending on its symmetry. But we could go further: if a hypercube is moved in a perpendicular direction a 5-cube is generated.





And we could continue doing the same, undoubtedly obtaining yet more beautiful images.

A NEW LANGUAGE FOR ARCHITECTURAL DESIGN

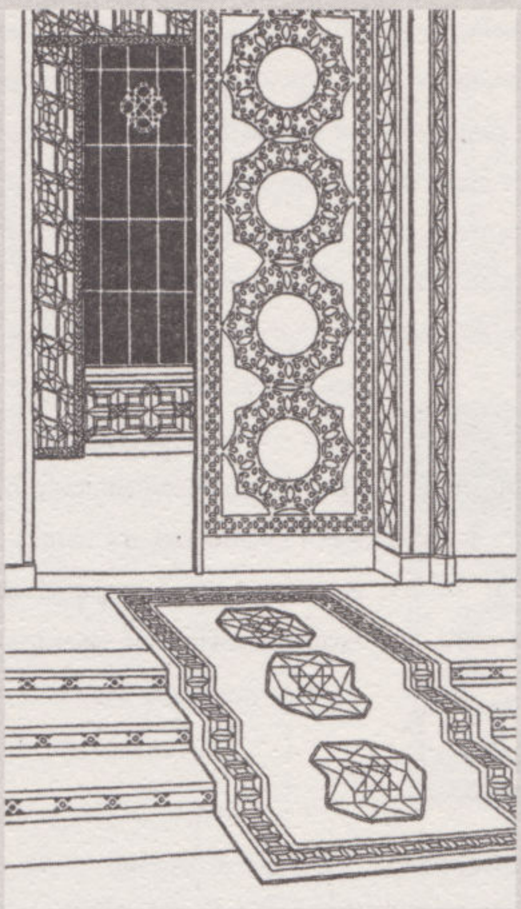


Illustration from the book Projective Ornament, *in which the fourth dimension is used in the design of decorative motifs.*

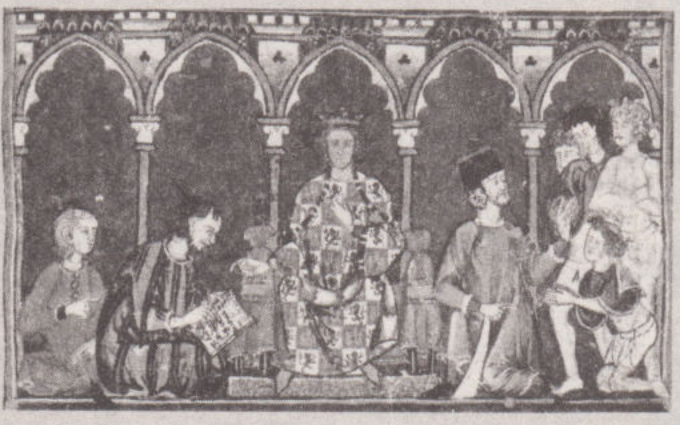
American architect, writer, designer and theosophist Claude F. Bragdon (1866-1946), in his book *Projective Ornament* (1915), introduced "projective ornamentation", a system for producing geometric designs to be used in architecture, graphic design or decoration. This method became widely used in modern architecture, where it was used in the Rochester Chamber of Commerce (1915-1917), as well as the design of magazines, posters and books. Bragdon suggested the need to create a new language for architecture and the ornamentation that was to be developed from geometry. Moreover, the fourth dimension is one of the fundamental tools for ornamental projection. He states that "the decorative motifs of the new aesthetic may appropriately be sought in four-dimensional geometry".

Perspective projection

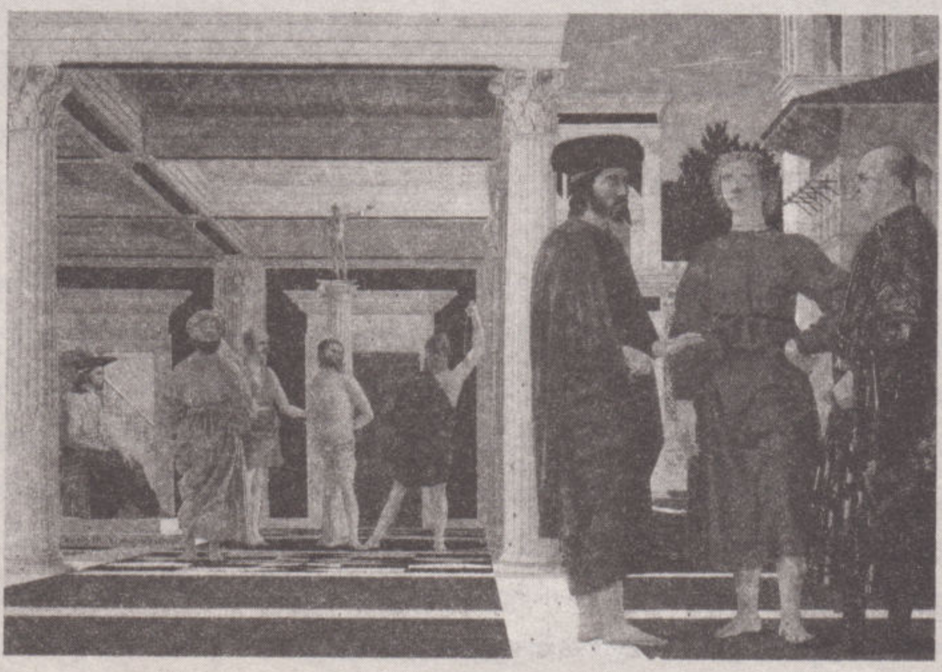
The images of the cube and the hypercube that we obtained in the previous section were 'shadows' produced by parallel 'rays of light'. However, now we are going to consider the shadow generated by a rod of light emanating from a point, source of light, and also the image our eye sees, or that which is captured by a camera. The corresponding projection is called perspective projection. It is the application of n-dimensional coordinate space in a $(n-1)$-dimensional subspace, which involves projecting from the light source, in other words, all the points that are on one straight line, which pass through the light source, will be projected onto the point of $(n-1)$-dimensional subspace which intersects that straight-line.

PERSPECTIVE IN ART

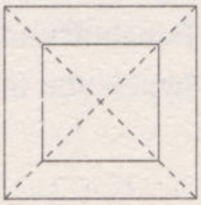
Renaissance perspective was a scientific and artistic revolution in representing space on a plane. In ancient times and the Middle Ages, images in paintings were flat, in the sense that they had no depth, proportions were not preserved and shapes and volumes were deformed. In the Middle Ages for example, more important people, from a religious point of view, were the largest figures. The Renaissance turned to science in the search for techniques and devices that allowed painting with perspective, to bring the image closer to the picture formed in the artist's eye. Among the great characters of perspective are Giotto, Piero della Francesca, Brunelleschi, Leon Battista Alberti, Rafael, Durero and Leonardo da Vinci. Renaissance perspective would dominate art from the 15th century until the 19th.

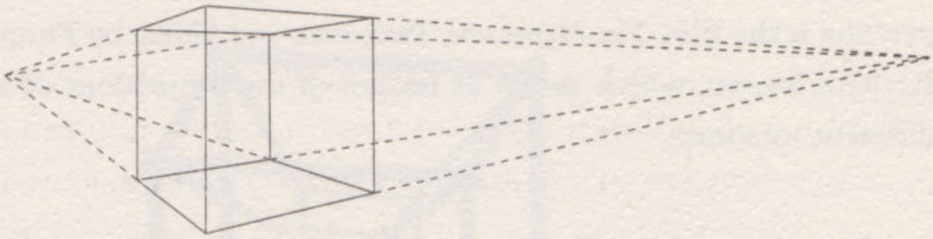


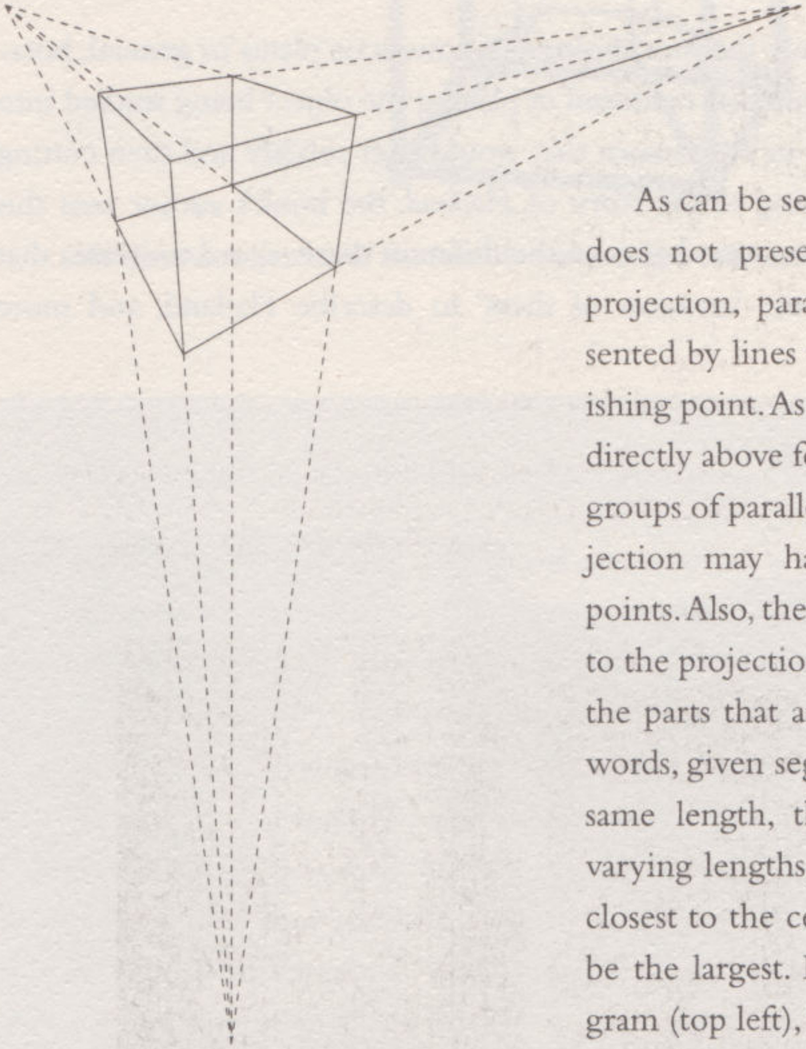
Above, a scene showing King Alfonso X the Wise and his court. This is an example of flat painting from the Middle Ages. Below, La Flagelación *(1444-1469), by Piero della Francesca. The Renaissance brought with it the development of linear perspective.*



If we now project a three-dimensional cube using perspective projection from three different exterior light sources, we can obtain the following images.







As can be seen, perspective projection does not preserve parallelism. For this projection, parallel lines will be represented by lines which converge at a vanishing point. As can be seen in the image, directly above for a cube, which has three groups of parallel lines (or sides), this projection may have 1, 2 or 3 vanishing points. Also, the image of the parts closest to the projection point appear larger than the parts that are furthest away; in other words, given segments of the cube of the same length, their images would have varying lengths, and that of the segment closest to the centre of projection would be the largest. Likewise, in the first diagram (top left), the exterior square represents the face closest to the light source, while the interior one is the image of one the furthest away.

Similarly to the perspective images of the cube we can obtain different perspective projections of the hypercube in our three-dimensional space, depending on the position of the projection light source in four-dimensional space. The following illustration is the representation of the hypercube represented in the same way as the first figure in the previous image. As in the three-dimensional case, the exterior cube represents the cubic face closest to the projection light source, while the interior one is the image of the furthest cubic face.

One of the most interesting documents regarding the visualisation of the hypercube is the film *The Hypercube: Projections and Slicing* by Thomas Banchoff and Richard Strauss, which shows us images of the projections of the hypercube in different rotations.

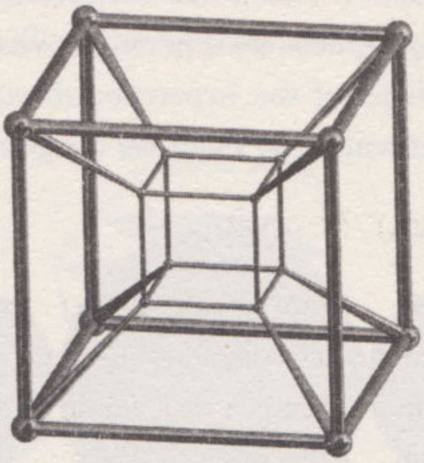
Cross-sections of the hypercube

In the past, in order to study the morphology of flowers or plants in general, botanists used to use a technique that consisted of placing the object being studied into a solid container, pouring in a substance that would later solidify and then cutting it into fine slices. Returning to the story of *Flatland*, the book's author uses this technique to transfer information between the different dimensional universes that appear. The square uses the 'creation of slices' to describe Flatland, and more

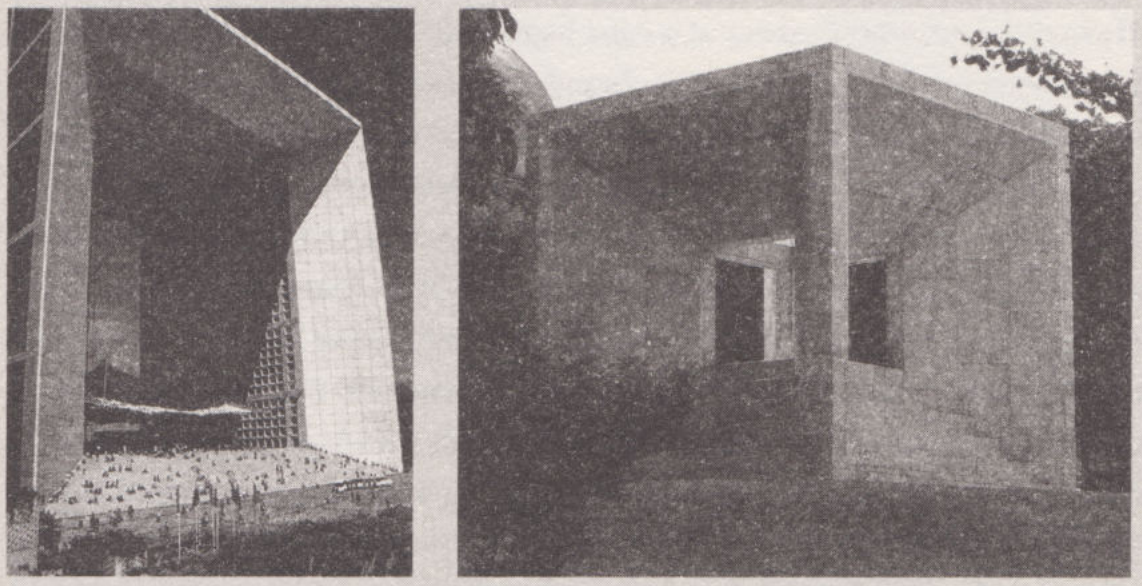
THE HYPERCUBE IN ART

The various representations of the hypercube, among them perspective projection, have captivated artists ever since the fourth dimension entered popular culture. Architects, painters and sculptors have made them a central theme for many of their works. An example of sculpture that uses perspective projection of the hypercube is the *Monumento a la Constitución*, which is in the gardens of the Museum of Natural Sciences of Madrid. Made from Andalucian white marble as a symbol of purity, its exterior cube has sides of 7.75 m, the four lateral faces are open and each one contains six steps that lead to the central cube, so that it can be accessed by the four cardinal points, a reference to democratic values. The hypercube represents a higher reality than our three-dimensional space, which corresponds to the values of the constitution: justice, liberty etc...We can find an architectural example of the hypercube in the *La Grande Arche de la Défense* in París. Built to a design by Danish architect Otto von Spreckelsen in 1989, this impressive building with sides of 110 m and shaped like a hypercube projected in perspective, contains a conference and exhibition centre, a computing museum and a viewpoint in its upper section, while the sides contains government offices.

specifically, to describe itself to the king of Lineland. In order to do this it transfers its body to the one-dimensional universe. Equally, the sphere, when crossing Flatland, to the complete confusion of the square, tries to explain to it the reality of the existence of the sphere itself and its universe. What is it that the square sees in its



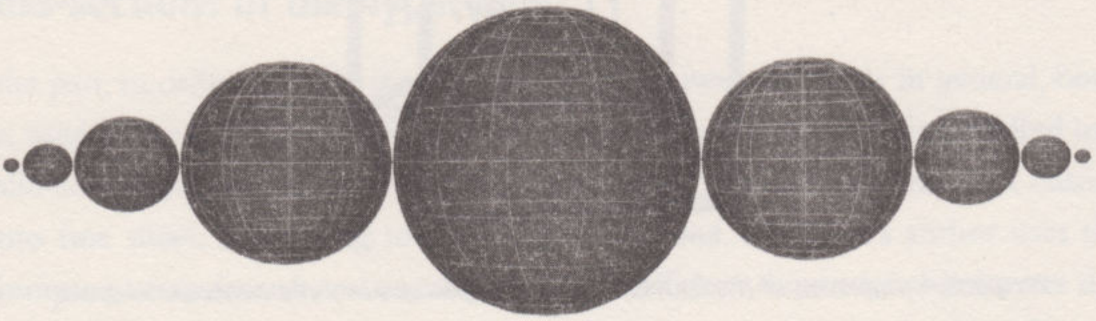
Perspective projection of the four-dimensional hypercube in three-dimensional space.



On the left, La Grande Arche de la Défense *in Paris, a hypercube which commemorates the 200th anniversary of the French Revolution. On the right, the* Monumento a la Constitución *(1979), designed by architect Miguel Ángel Ruiz Larrea, which uses perspective projection of the hypercube.*

world when the sphere travels to Flatland? First it sees a point, and then, immediately afterwards, a circle (could it be a Flatlander priest?), which grows and then diminishes until becoming a point again and disappearing. We would see the same if we were visited by a hypersphere, although in place of a circle we would see a sphere changing size. In other words, three-dimensional slices (now our knife is three-dimensional) of a hypersphere are spheres that change in size.

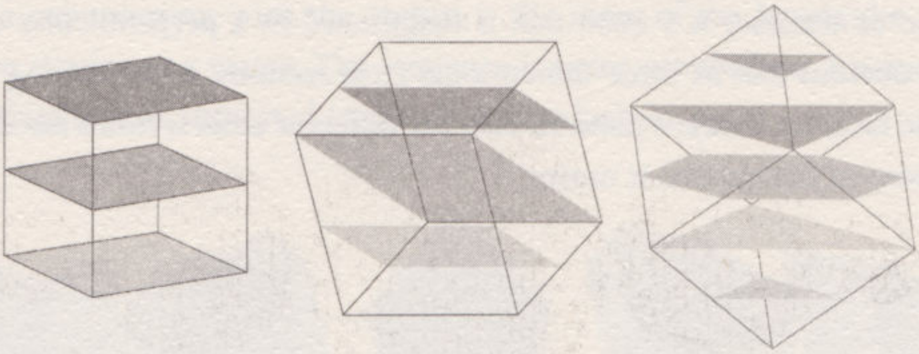
Before analysing the shape of the hypercube through the three-dimensional slices obtained by cutting it with a 3D knife, we are going to study the case of one dimension less, in other words, the flat slices obtained by cross-sectioning a cube in different directions, in order to use in the dimensional analogy later.



Three-dimensional sections of a hypersphere (graphics by Josu Arroyo).

If we make flat cross-sections of a cube beginning with one of its faces, in other words, making parallel cuts through it, then the sections obtained are squares, as can be seen in the diagram below. If cross-sections are made starting the cutting at one of the cube's edges, with planes perpendicular to the plane that joins that section to its opposite, then the slices obtained are line, rectangle, square, rectangle and line again. The most interesting sequence of cross-sections and, at the same time, the most difficult to visualise, is the one obtained when flat slices are made starting the cuts on one of the vertices, with planes perpendicular to the diagonal that joins this vertex to its opposite. Starting with the vertex the cross-sections are a triangle that increases in size and then another one that decreases until it disappears at the opposite vertex. But what shape would we see in the middle. It may seem surprising, but it is a regular hexagon, in other words, one whose sides and angles are equal.

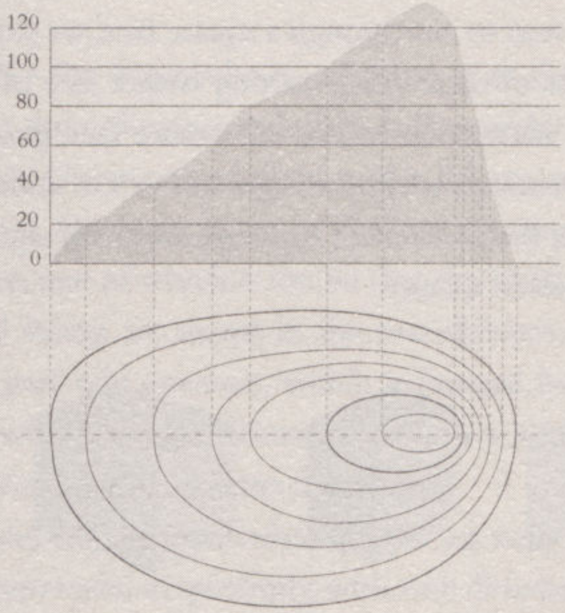
The reason is this: We have triangles until their vertices meet three of the cube's vertices, at which time the triangles are truncated, forming an irregular polygon with six sides (with sides of different lengths), which continues to change until becoming a triangle again, which decreases in size. But now these triangles have one of their vertices oriented in the opposite direction to the vertices of the first triangles, and so due to symmetry at the middle point we have a hexagon.



Flat sections of a cube depending on the direction of the cut.

CONTOUR LINES

An everyday use of flat sections of a three-dimensional object in order to obtain information on its geometry, its shape, are the curved contours that appear on topographical maps and represent the points that are at the same height above sea level. They show horizontal planes at various heights cutting the surface of the terrain into planes. The curves appear where these planes intersect the surface. If they are very close to each other it means that there is a steep slope in the terrain, while if they are far apart, the ground it is more level. This, together with the use of colours on topographical maps, gives us very interesting information on the relief.

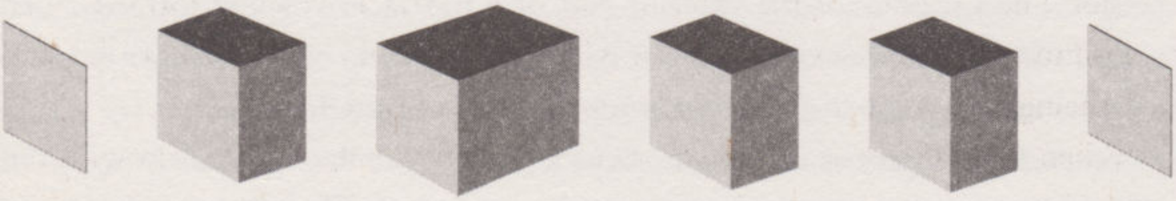


Contour lines tell us about the relief of a piece of terrain.

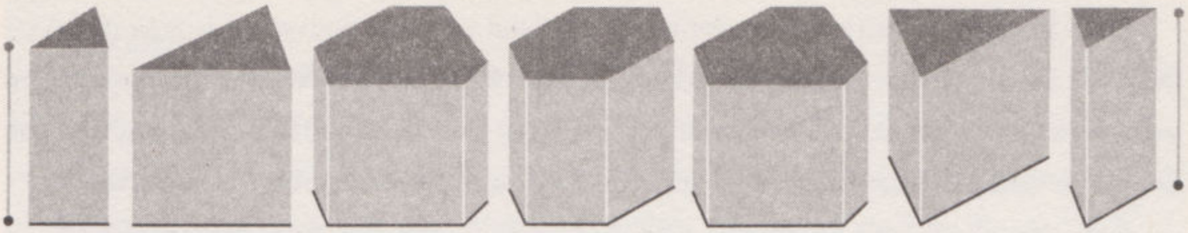
To make three-dimensional slices of the hypercube we will need our knife to be three-dimensional. And also, as with the case of the cube, we will cut slices starting with a cubic face, then a square one, then a side and a vertex. It should be remembered that we can also think of the hypercube as falling through our three-dimensional space and we are going to study what we see based on the first part of the hypercube we see.

If we take into account, as we have mentioned throughout the book, that a hypercube or tesseract is a cube that is lengthened in a perpendicular direction, then it is obvious that its three-dimensional cross sections, when cutting is started on a cubic face, are always a cube. In fact, the different cross sections are the different positions of the cube as it moves.

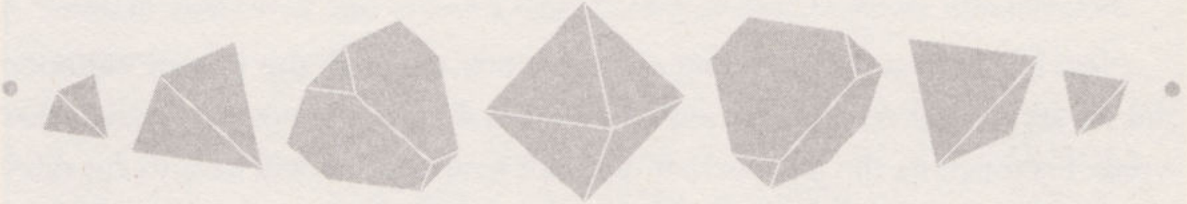
In order to understand what the cross-sections of the hypercube are like when we start to cut through a square face, we simply need to remember the shape of the slices of cube when starting with a face of this type or an edge. As can be seen in the illustration below, the square face generates square cross sections during the movement, while the segments of the initial square face generate rectangles; therefore, the sections of the hypercube are square at the ends with rectangular prisms between them.



The cross sections of a cube from one side and one vertex explain the shape of the three-dimensional cross sections of a hypercube when we start cutting at an edge. The sequence of 3D slices would be line, triangular prism, irregular hexagonal prism and regular hexagonal prism, and these shapes repeat in the opposite order.



The most interesting case, which can be compaed with cutting the cross sections of a cube starting with the vertex, is the slices of the hypercube produced when also starting at a vertex. The sequence is point, tetrahedron, truncated tetrahedron, octahedron, truncated tetrahedron, tetrahedron, point.

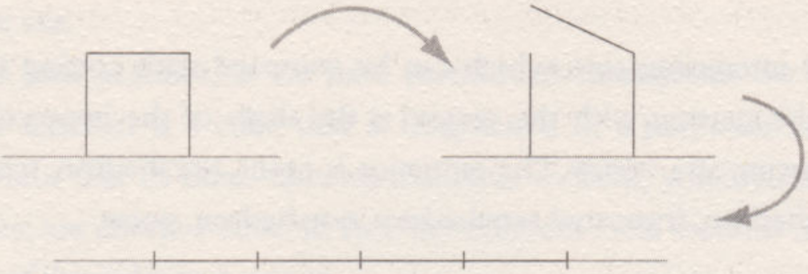


Unfolding the hypercube

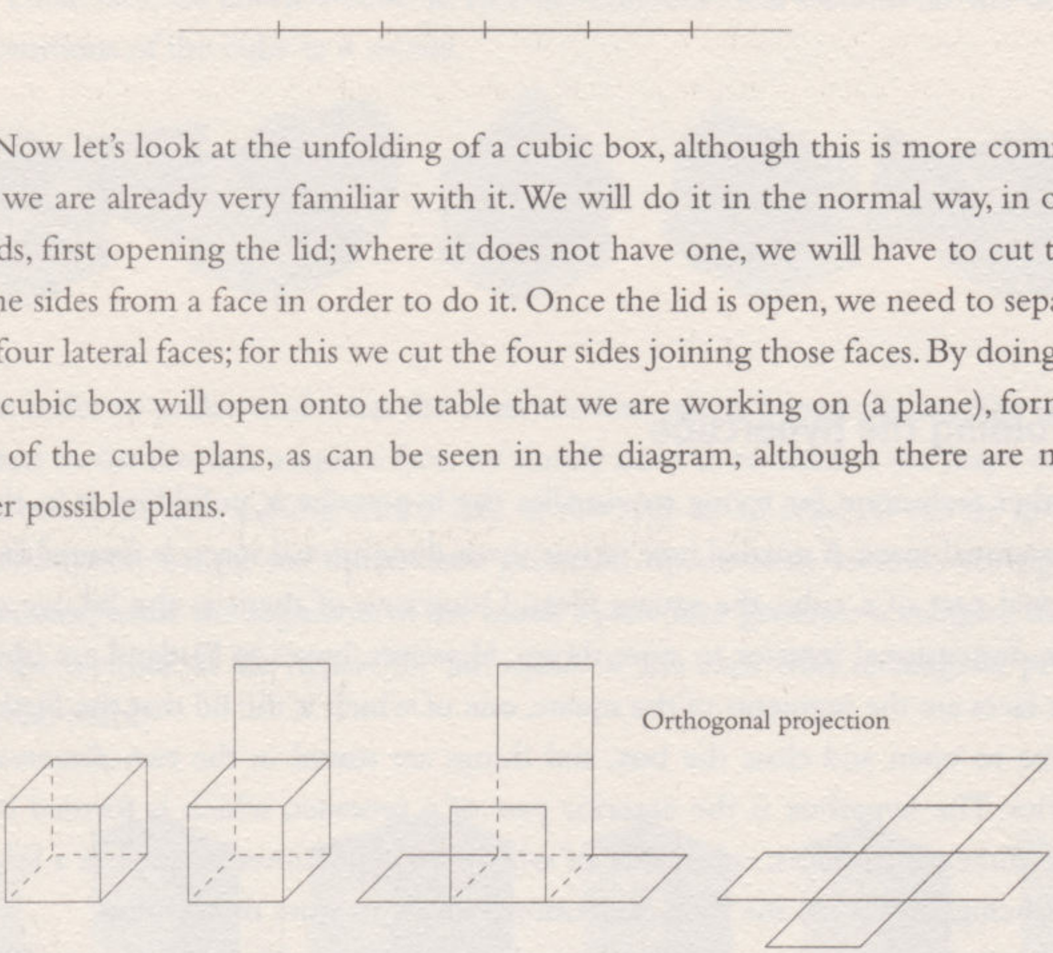
Another technique for trying to visualise the hypercube is unfolding it in three-dimensional space. A normal box, in our three-dimensional space, is formed by the external part of a cube, the square faces. Using one of them as the lid, we get a three-dimensional interior to store things. However, boxes in Flatland are squares, their faces are the segments of the square, one of which is the lid that the Flatlanders use to open and close the box, and things are stored in the two-dimensional interior. The hyperbox is the exterior part of a tesseract, which is formed by its three-dimensional faces, cubes, one of which will sometimes be used as a lid, and hyperbeings could use the four-dimensional inside to store their things.

When we unfold a square box, or cube, or a hypercube, we end up with the external parts of a square, a cube or a hypercube, respectively, with one dimension less and, therefore, we can unfold it into a space of fewer dimensions. The square box in Flatland can be unfolded in Lineland, and maybe its king could use it to understand what a square is. The normal, cubic box can be unfolded in a plane, helping the square to understand the form of a cube. And, finally, we can unfold the hyperbox in our three-dimensional space, allowing us to better understand what a hypercube is. Overleaf, we will see what we end up with in each case.

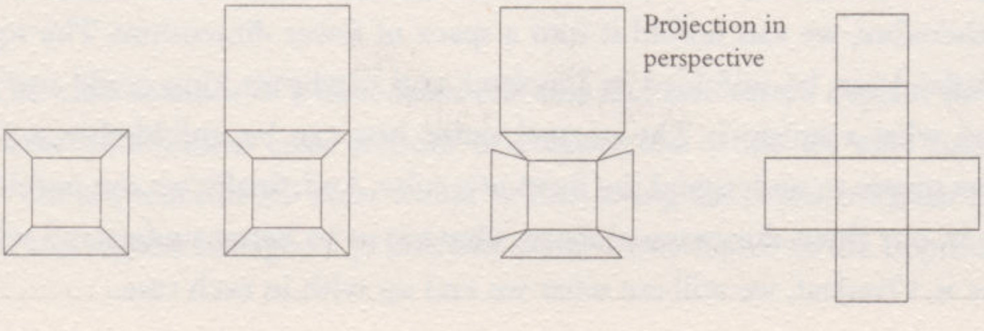
Let's imagine that the protagonist of *Flatland* unfolds one of its boxes in Lineland. To do so, first it opens the lid of the box (if it doesn't have a lid, two of the sides need to be separated by breaking a vertex) and then unfolds it laying out the faces. We end up with four equal sections, in line with each other, in Lineland.



Now let's look at the unfolding of a cubic box, although this is more common and we are already very familiar with it. We will do it in the normal way, in other words, first opening the lid; where it does not have one, we will have to cut three of the sides from a face in order to do it. Once the lid is open, we need to separate the four lateral faces; for this we cut the four sides joining those faces. By doing this our cubic box will open onto the table that we are working on (a plane), forming one of the cube plans, as can be seen in the diagram, although there are many other possible plans.



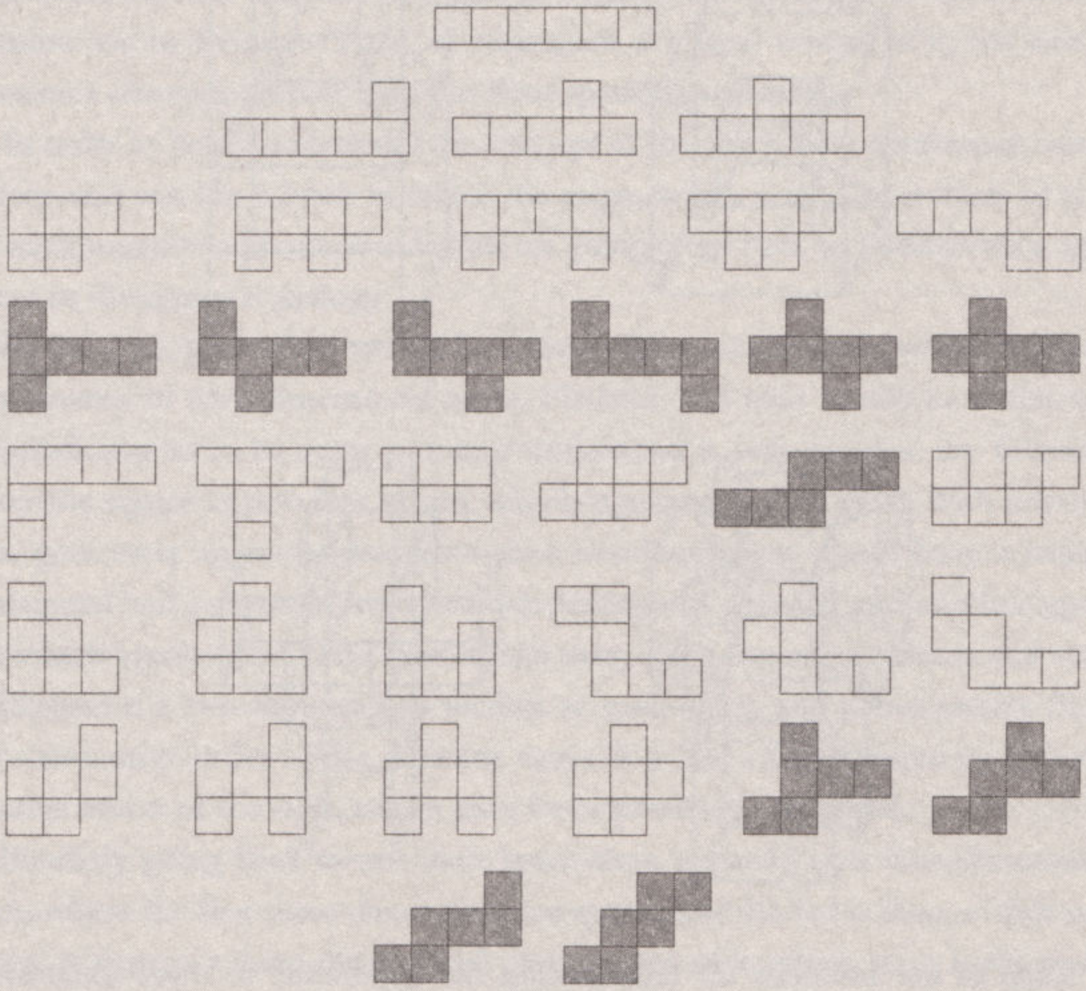
Orthogonal projection



Projection in perspective

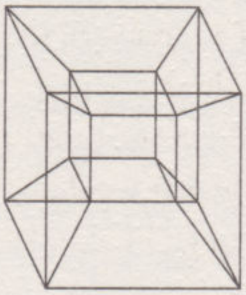
HEXOMINOES

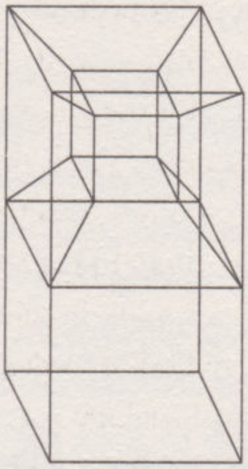
Flat shapes formed by joining six squares edge to edge (they cannot be connected only by the vertexes) are called hexominoes, an example being a cubic box unfolded onto a plane. Finding out how many exist is an interesting problem. One way of reaching the solution is starting with the number of squares the figure has. In general, a grade *n* polyomino is a flat shape formed by joining *n* squares joined by their edges. There is only one domino ($n = 2$). Adding just one square, 2 triominoes can be formed ($n = 3$). With one more square we get exactly 5 tetrominoes, which, by the way, are the shapes that appear in Tetris. There are 12 pentominoes with which a very interesting learning game can be made. Finally, adding one square more to those 12 pentominoes, we can see that there are 35 hexominoes. But how many of them can be used to form a cube? Coming up with the answer is an fun game. Have a go!

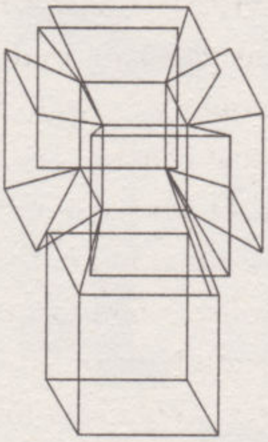


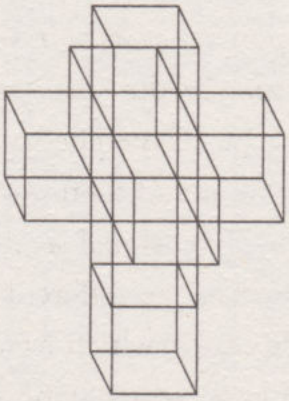
The 35 possible hexominoes, and in black the 11 which can form a cube.

Now we are ready to understand how to unfold the hypercube, using the dimensional analogy of the two cases analysed to do it. As with them, we take the upper lid of our hyperbox, in other words, one of the faces, which we know is cubic. To open it we have to free its upper lid (which is cubic), which is joined to the other six faces of the hypercube, and to do this we cut through the squares in contact with five of these cubic faces with our upper lid-face. Now the hypercube is open and we need more cuts to unfold it. The squares that join the six cubes that were in contact with the upper lid are cut (take note that there are eight cuts). At last we have the hypercube unfolded in our three-dimensional space.









Each of the methods for representing the hypercube in three-dimensional space shows us part of the information on our four-dimensional object, but it also hides another part from us, and even produces some distortion. The mathematical projections distort the hypercube, but they keep all the information on the spatial relationships in the fourth dimension of the elements that make it up. The cross sections hide a lot of information, since each of them is a very small part of the object, but there is no distortion, and the sequential study of the slices also offers useful information on the hypercube's internal structure. Their unfolding shows us, without distortion, the elements of the hyperbox, but we have lost the four-dimensional nature, and we have to imagine the space in which that hyperbox is enclosed.

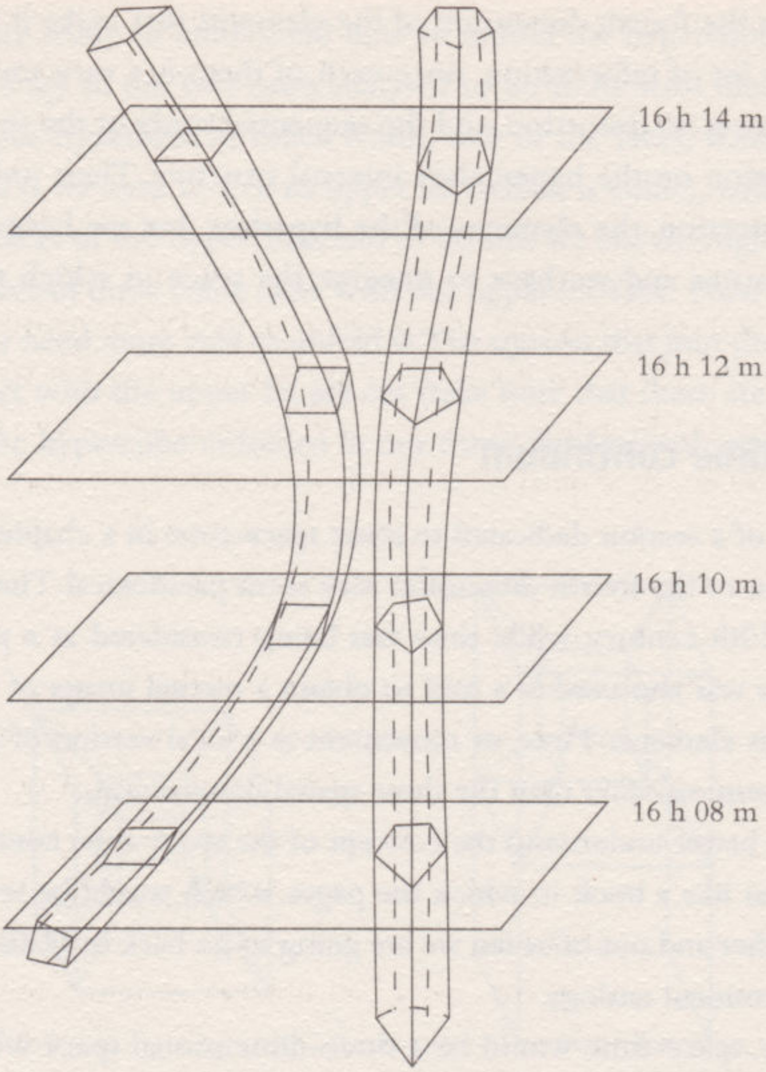
The space-time continuum

The inclusion of a section dedicated to static space-time in a chapter dedicated to the visualisation of the fourth dimension may seem paradoxical. The reason is that when, in the 19th century, while time was being considered as a possible fourth dimension, this was also used as a tool to obtain a mental image of the fourth dimension and its elements. Time, or movement as a local version of it, was used to generate a dimension other than the three spatial dimensions.

In order to better understand the concept of the space-time continuum, which Hinton said was like a book in which the pages, which would be sections of time, are stuck together and out of order, we are going to go back to Flatland once again to use its dimensional analogy.

In this case, space-time would be a three-dimensional space where the space part consists of two-dimensional space, Flatland, and time would be a direction perpendicular to it. In order to better understand it, let's imagine the following scene: the square approaches its son, which is pentagonal, to speak with him and, then goes away again. In the space-time continuum we would have a straight pentagonal 'rail', a type of stretched-out pentagonal column, and another square rail, which approaches it and then moves away. Each instant of time in our scene in Flatland is a two-dimensional section of space-time, and consequently, this is the consecutive order of the different times. It is like a film formed by the consecutive union of the stills, which have been previously recorded.

Similarly, going back to our static space-time, it would be a four-dimensional space, where the first three dimensions are spatial, and the last is time, so that each instant of time is a three-dimensional cross section of it, which to us looks similar to a spatial four-dimensional space. In it, we are no more than finite rails within space-time. Static space-time is cohabited by the past, the present and the future; but, why is it not possible to see the past or the future, even though they exist?



For Flatland, time-space is a three-dimensional space, where the square, or its pentagonal son, are shown as rails which get longer with time.

Furthermore, what makes our perception of the passing of time constantly move forward? There are people who think that this is a property of our universe and it has to be accepted as it is. For example, in his article *The Myth of the Passage of Time* physicist David Park says "...all instances of our lives, of our history, coexist, and the illusion of the passing of time is a property of the universe that is observable but cannot be explained..."There are those who think that the passing of time is something subjective which happens in our minds, and that it is possible to reach a mental state in which we may change the location of our conscience in space-time, although this idea has few proponents. For many, a film or a book is a metaphor for the space-time continuum, since they exist in our collection even though we may not be watching or reading them, and the perception of the passing of time would be the watching of the film or reading of the book. But this throws up some interesting questions: is it possible to see the film several times, or even infinitely? Is it possible that at a certain point of the film we have the opportunity to return to a previous point or advance to a subsequent one? If this is so, where is time's remote control?

Another interesting question linked to the theory of static space-time is the problem of free will. A common objection to it is that the future does not seem to be completely determined by that which occurs immediately beforehand, and even if it were predetermined that does not imply that it can be predicted. If it were then that would cost the validity of the space-time continuum. Apparently, it would mean that we do not have freedom to act, that our paths are marked out before we are even conscious of the fact that we have begun travelling along them. When something happens, or even when someone does something, we tend to ask ourselves, how did that happen? What caused that event or motivated the action? We normally think, although we may believe in free will, that there are always internal or external reasons that explain any event. For example, when we study physics we get the impression that there is a reason for everything, and nothing is random; the position of a particle is determined by its initial conditions, the space and the forces acting on it. However, physics has demonstrated that there are events that may or may not occur, without dependence on that which happened in the previous instance, for example, in the case of an atom of uranium disintegrating and emiting radiation. A solution to the problem of free will in the space-time continuum, based on what we showed above, is 'parallel universes', an idea suggested and studied by renowned physicists such as American Bryce DeWitt in his *The Many-Worlds Interpretation of Quantum Mechanics*. In the parallel universe model, all possible universes coexist, each one joined to others in the form of branches, and our universe is nothing more than one of the possible paths. In every instant an undetermined quantum event will take place or not take place in an atom; then the universe, or rather, its branch, will divide into two. This means a huge number of new branches every second, and an infinity of parallel universes. There will be universes in which we exist and others in which we do not, and they may even be universes where we have four arms or can fly.

Chapter 8

The fourth dimension in 20th-century art

In plastic art, I believe, there is a fourth dimension, which may be described as the consciousness of a great and overwhelming sense of space-magnitude in all directions at one time, and is brought into existence through the three known measurements. It is not a physical entity or a mathematical hypothesis, not an optical illusion. It is real, and can be perceived and felt.

Max Weber, The Fourth Dimension from a Plastic Point of View *(1910).*

Renaissance perspective, which sought to represent what was seen by the artist's eye, the human eye, dominated five centuries of art. However, the invention of the photograph, which made it possible to obtain a true perspective image of what lay before the camera, together with a range of other philosophical, social and cultural factors, meant that artists increasingly moved away from Renaissance perspective. At first it was timid, with the Impressionists, until reaching a complete rupture, started by the French painter Paul Cézanne (1839-1906), but which really came into its own with Cubism.

Non-Euclidean geometries and multidimensional geometry contributed to the Cubists' complete break with perspective. Furthermore, from then on, the fourth dimension would become present in practically all the avant-garde movements of the 20th century. Regardless of whether from a geometrical, philosophical or poetic point of view, it captivated the artists who became interested in it and made use of it from their own perspective.

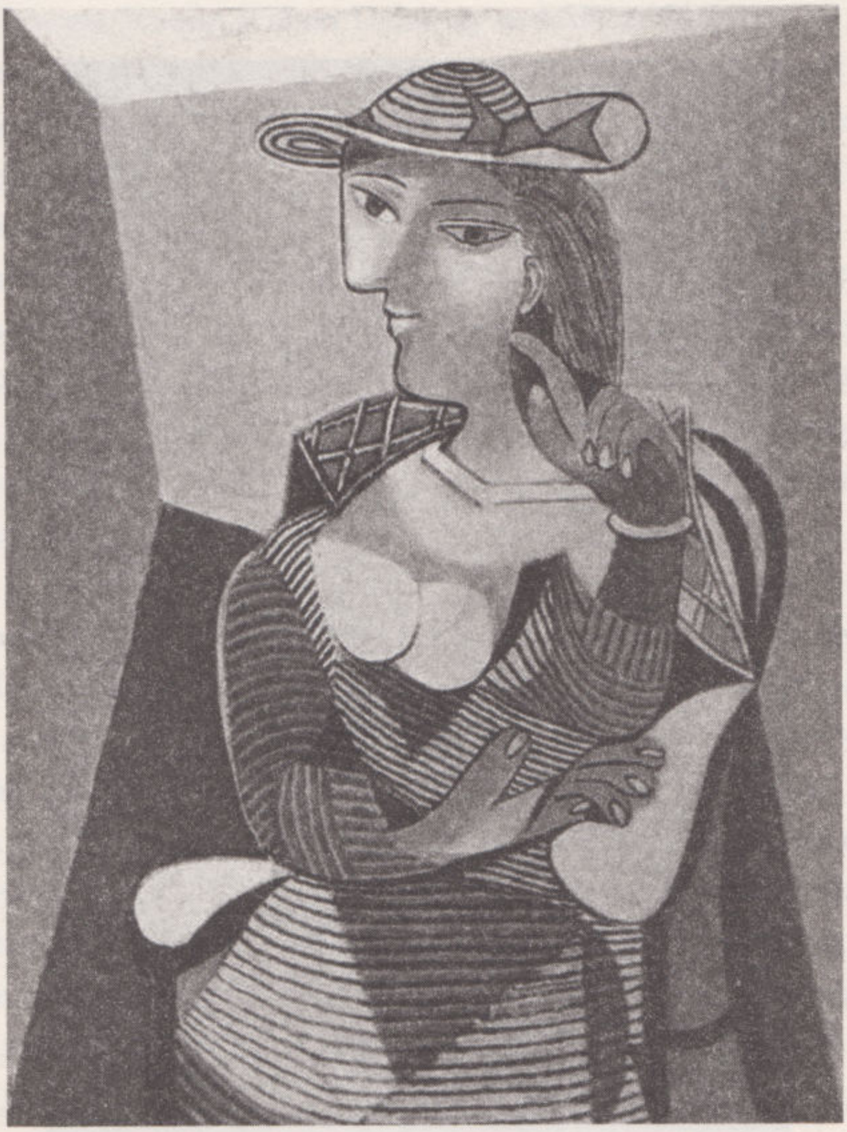
Many mathematicians complained about the lack of scientific rigour in the artists' treatment of the fourth dimension, yet how can we reject their interest in it, or in any other mathematical topic, when they work beyond science itself? Moreover, is this not normal for the artist's mode of operation, their way of thinking and creating, of reflecting on social and personal matters, forging their own path in order to later manifest it through their art? Why is such an action, considered acceptable for other subjects, not also valid for mathematics? If artists are not practising mathematics, why should we demand scientific rigour from them? Their interest in the universe, inside and outside, must lead them to every corner of the world, including mathematics. We cannot force them to use scientific method but should allow an 'artistic method'. Artists add the fourth dimension, non-Euclidean geometries and multidimensional geometry to their baggage and start out on their own journey with the contents.

Cubism and the break with perspective

The fourth dimension was a symbol of liberation and a source of new ideas for the artists of the time, particularly for the Cubist movement. Enthusiastic about the existence of higher dimensional spaces, a higher reality, they attempted to break with Renaissance perspective, to escape from the 'visual reality' marked by the vision-projection of an eye in three-dimensional space. The fourth dimension also contributed to the development of an idealist philosophy at the end of the 19th century, which went against the prevailing positivism.

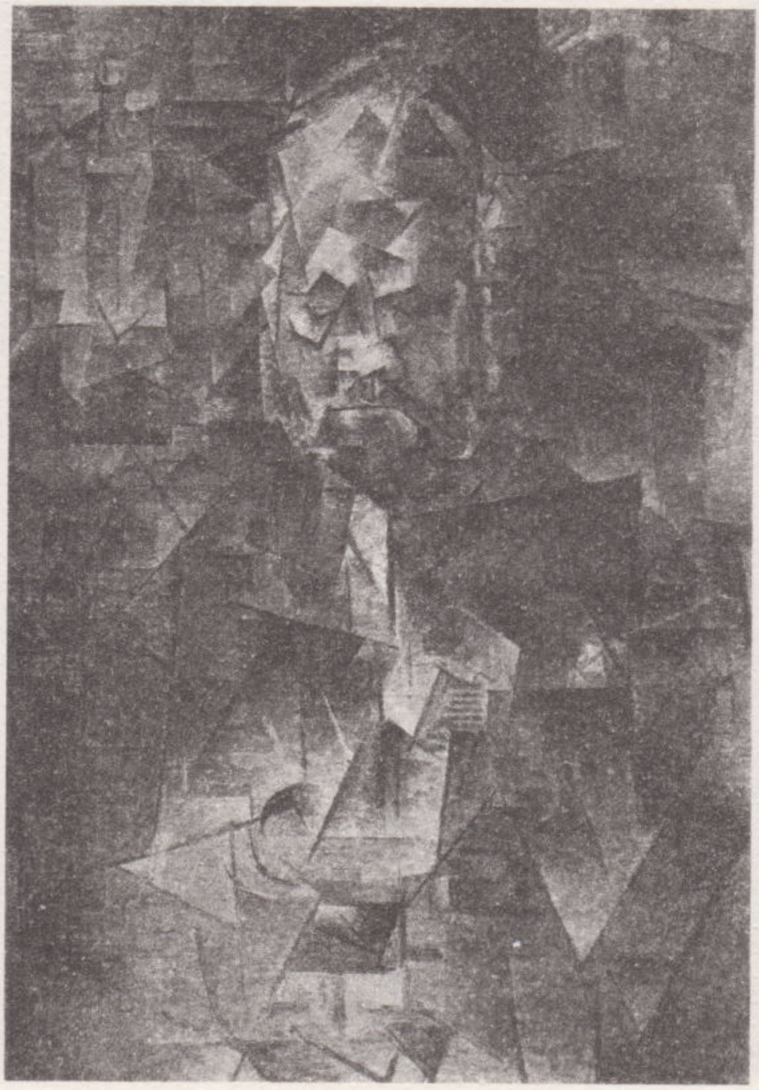
Two of the most theoretical Cubists, the French painters Albert Gleizes (1881-1953) and Jean Metzinger (1883-1956), explain in their book *On Cubism* that the Cubist painter, in contrast to the Renaissance one, does not attempt to give expression to an object as he or anyone else sees it, but attempts to represent it as it is. There is an interesting anecdote that illustrates the subject, in which Pablo Picasso (1881-1973), who was born in Malaga, is travelling on a train. A passenger recognises him and decides to ask him why he can't just paint people as they are in reality, why does he have to distort their images? Picasso asks the man for a photograph of his family. After observing the photograph, Picasso replies to him: "Oh, and is your wife really small and flat?". In short, the perspective image of an object or scene, no matter how realistic the painting, even if it is a photograph, does not show the object as it is, but a perspective of it. In an article that appears in the journal *Comœdia Illustré* (1913), the critic Maurice Raynal writes: "Instead of painting objects as they saw them, primitives painted them as they thought they saw them, and this is precisely the law which Cubists have repurposed, expanding it and classifying it under the name of the fourth dimension."

The Cubists' solution for breaking with perspective and trying to paint the object as it is and not as it is seen, is to introduce different points of view to the canvas. We can find such an example in Picasso's painting, *Portrait of Marie-Thérése Walter* (1937), which despite not being considered as part of his Cubist period (which according to the art historian Douglas Cooper dates from 1907 to 1921) shows us different perspectives in the same painting.



In the painting Portrait of Marie-Thérése Walter, *Picasso paints his partner from various perspectives. The hat shows us two different perspectives: one above it and the other below. The face of Picasso's lover (in classic Picasso style) requires a point of view for each of its eyes, another for the lips, a fourth for the nose and yet another for the hair. In the body, we can also make out more than one perspective. The chair requires at least two points of view, as shown by its arms. The floor positions the observer above the women, looking down from above, while the roof is painted as if seen from below.*

However the technique used during the initial years was to break the figure into small planes or faces, each of which showed a different point of view, and which together formed the image of the subject expressed on the canvas, the intention being to represent it from all points of view. Thus, the image acquires a certain complexity which recalls the representations of four-dimensional objects. We can find an example of this Cubist technique in paintings from Picasso's Analytic Cubism period, such as *Portrait of Ambroise Vollard* (1910).



In the painting Portrait of Ambroise Vollard, *Picasso decomposes the art dealer into multiple planes showing multiple perspectives.*

Another example is Metzinger's *Nude* (1910). Metzinger was one of the most enthusiastic painters when it came to the fourth dimension and in this piece, multiple points of view create a highly complex image which manages to break with any notion of the figure's three-dimensional existence, even suggesting its four-dimensional reality. Other works by Metzinger in which the viewer can see multiple points of view, although which are not as complex as the previous example include *Tea Time* (1911) and *Dancer in a Café* (1912). However, *Woman with Hat* (1913) shows a completely different style, with different perspectives of a woman's face on vertical planes. Picasso and Metzinger were the two Cubists who pushed the multiple plane technique farthest, although it is also present in the work of artists such as Georges Braque, Robert Delaunay and Albert Gleizes.

In Chapter 4 we mentioned that the retina of a hyperbeing would simultaneously capture the outer surface of an object as well as its inside, or rather it would see us from all possible points of view. Broadly speaking, this was also the Cubists' goal. Additionally, some of them made use of thought or the imagination as a means to lead them to the fourth dimension, as is shown by the following Picasso quote, "I paint objects as I think them, not as I see them". Cubist painters associated the space of Renaissance perspective with Euclidean three-dimensional space, while they associated the new space of the painting they introduced with the fourth dimension and non-Euclidean geometries. Thus in their work *On Cubism,* Metzinger and Gleizes explain:

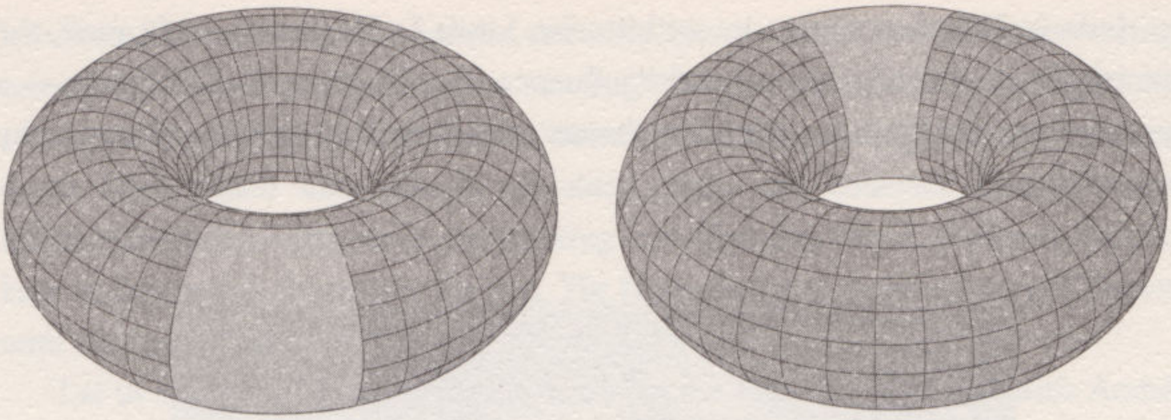
> "Let us admit that the reminiscence of natural forms cannot be absolutely banished; not yet, at any event. Art cannot be raised to the level of a pure effusion at the first step. This is understood by the Cubist painters, who indefatigably study pictorial form and the space which it engenders.
>
> This space we have negligently confounded with pure visual space or with Euclidian space. Euclid, in one of his postulates, speaks of the indeformability of figures in movement, so we need not insist upon this point.
>
> If we wished to relate the space of the painters to geometry, we should have to refer it to the non-Euclidean mathematicians; we should have to study, at some length, certain of Riemann's theorems."

The poet and critic Guillaume Apollinaire (1880-1918) shows a similar way of thinking in *The Cubist Painters. Aesthetic Meditations* (1913):

> "[Chapter II] The secret aim of the young painters of the extremist schools is to produce pure painting. Theirs is an entirely new plastic art. It is still in its beginnings, and is not yet as abstract as it would like to be. Most of the

> new painters depend a good deal on mathematics, without knowing it; but they have not yet abandoned nature, which they still question patiently, hoping to learn the right answers to the questions raised by life.
>
> [Chapter III] The new artists have been violently attacked for their preoccupation with geometry. Yet geometrical figures are the essence of drawing. Geometry, the science of space, its dimensions and relations, has always determined the norms and rules of painting. Until now, the three dimensions of Euclid's geometry were sufficient to the restiveness felt by great artists yearning for the infinite.
>
> The new painters do not propose, any more than did their predecessors, to be geometers. But it may be said that geometry is to the plastic arts what grammar is to the art of the writer. Today, scientists no longer limit themselves to the three dimensions of Euclid. The painters have been led quite naturally, one might say by intuition, to preoccupy themselves with the new possibilities of spatial measurement which, in the language of the modern studios, are designated by the term: the fourth dimension.
>
> Regarded from the plastic point of view, the fourth dimension appears to spring from the three known dimensions: it represents the immensity of space eternalising itself in all directions at any given moment. It is space itself, the dimension of the infinite; the fourth dimension endows objects with plasticity."

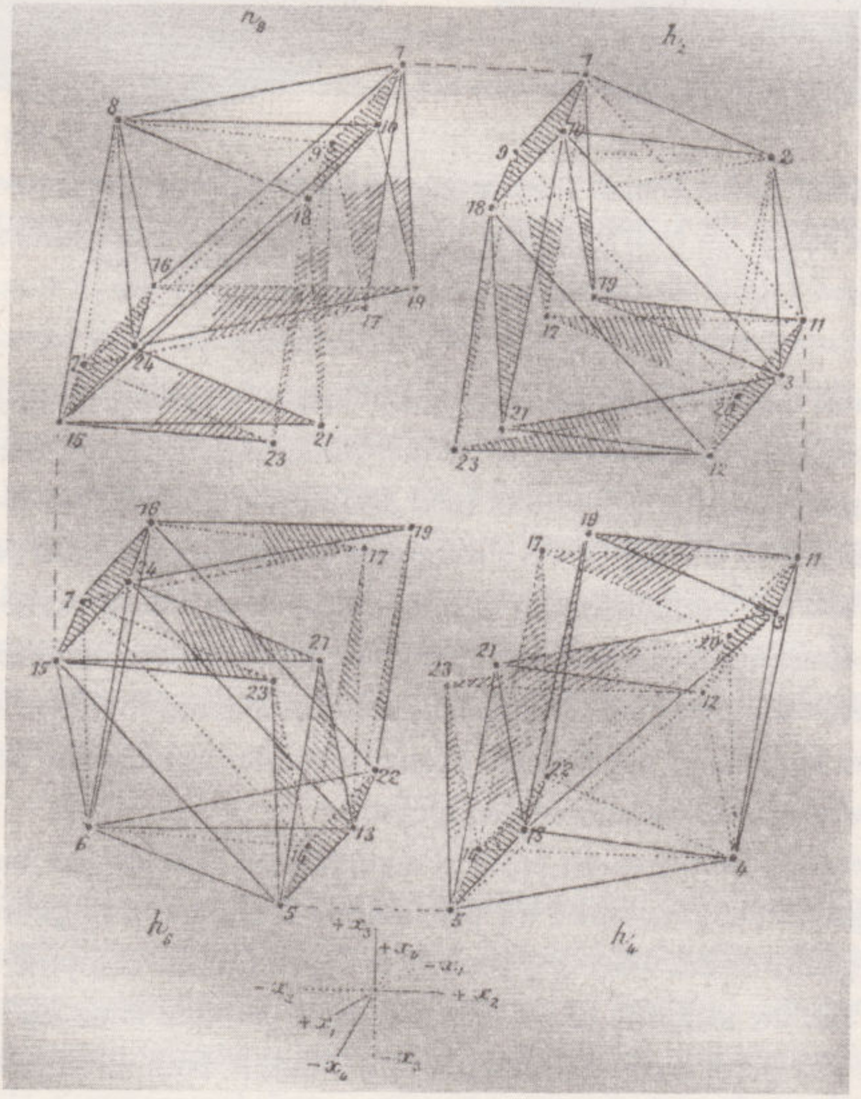
The first of these two quotations talks of Euclid and the "indeformability of figures in movement". This means that if we move a square on the plane (here we understand the term movement to mean displacement or rotation) it is not deformed. However, Riemann introduces spaces (either surfaces or multidimensional spaces) with variable curvature and thus as a figure is moved across the surface, its shape changes. Let us consider an example. In Chapter 3, it was explained that the surface of a rubber ring, the torus, has variable curvature which is positive on the outside and negative on in the inside. Thus, if we take a figure on the outside with a more or less rectangular shape, this will be convex in any direction. However turning it and moving it towards the inside, the figure will become distorted and will result in a different shape, convex in some directions and concave in others.



The "rectangular" area on the outside of the torus always curves the same way, however if it is moved over the torus, the rectangular area on its inside – the image has been rotated to make it possible to see this part – which curves in the opposite way.

Apart from the break with perspective, for Cubists, the fourth dimension meant a certain freedom in space and form. Further more, Metzinger and Gleizes, Apollinaire, and the Polish-born American artist Max Weber (1881-1961) all linked the fourth dimension to infinity. It was a somewhat metaphorical link, since for them, perspective and the third dimension were imprisoning art and its expression, while the fourth dimension set them free; hence the relationship with infinity. Albert Gleizes stressed this in an interview given in 1912: "Beyond the three dimensions of Euclid we have added another, the fourth dimension, which is to say, the configuration of space, the measure of the infinite."

Let us return however, to the point of departure for Cubism and the 20th century avant-gardes with Picasso's 1907 painting *Les Demoiselles d'Avignon (The Young Ladies of Avignon)*. In this work it is possible to appreciate the complete break with perspective and the inclusion of multiple points of view, thus initiating a new visual language. The piece is clearly influenced by Cézanne, both in terms of its break with perspective and its attempt to represent reality by reducing it to its essential forms. (Cézanne's aim was to "treat nature by the cylinder, the sphere, the cone".) Picasso came into contact with the fourth dimension through the mathematician and actuary Maurice Princet, who met the painter in 1905 though a shared lover and became a fringe member of the "Picasso Group". Princet introduced him to the work of Poincaré and Jouffret, and spoke to the members of the fourth-dimension group. In his study on Picasso and Einstein, Arthur I. Miller notes the similarities between the sketches for this painting with Jouffret's geometrical figures (see the illustration overleaf); and the same can be said for the aforementioned work *Portrait* *of Ambroise Vollard,* as shown by art historian Linda D. Henderson. Of course, the fourth dimension is only one of the influences for *The Young Ladies of Avignon*, a piece which also hints at many other themes, such as African painting, photography and sex.



Perspective Cavalière of the Sixteen Fundamental Octahedrons of an Icositetrahedroid, *from the book* Traité Élémentaire de Géométrie à Quatre Dimensions *(1903), by Jouffret.*

Yet how was this work, one so important to 20th-century art, received by the group of people surrounding Picasso? He unveiled the painting to them in his study and, by all accounts, it was very badly received. Braque, who would go on to become one of the Cubists closest to Picasso, joked that he had perhaps been drinking.

Leo Stein, Picasso's long time-patron, sarcastically remarked: "You have been trying to paint the fourth dimension; how amusing!" Matisse thought the painting a joke and swore to destroy Picasso. Derain claimed that the painting was the point of death, after which only suicide remained. Only the collector Daniel-Henry Kahnweiler took an interest in the painting and offered to buy it. However, Picasso kept it, saying that it was not for sale. *The Young Ladies of Avignon* was not shown until nine years later.

Let us end with a curious remark made by the French writer and critic André Salmon in 1912 and reprinted by Miller: "*[The Young Ladies of Avignon]* are naked problems, white numbers on the blackboard. It is the principle posed of painting = equation… Painting, henceforth, became a science and not one of the least austere."

The friendship between the French painter Georges Braque (1882-1963) and Pablo Picasso gave rise to Analytic Cubism, which began to attract other adherents, including Metzinger, Gleizes and Apollinaire. Then came the introduction of Cubism through the exhibition at the Salon des Indépendants in 1911, which included works by Metzinger, Gleizes, Henri Le Fauconnier (1881-1946), Fernand Léger (1881-1955) and Robert Delaunay (1885-1941), although curiously not by Picasso or Braque. The essence of Analytic Cubism would remain within the group of Cubists known as the Puteaux Group, while Braque and Picasso would go on towards Synthetic Cubism, dominated by collage.

The Puteaux Group, also known as the Section d'Or (Golden Section), in reference to the geometric ratio, was born following the decision of a group of artists,

PRINCET, THE MATHEMATICIAN OF CUBISM

Maurice Princet (1875-1973), a mathematician who worked as an actuary, was a figure of some importance among Cubists, earning him the title of "The Mathematician of Cubism". After making contact with Picasso, he joined the members of the Picasso Group and later the Puteaux Group. He often gave informal talks, especially on the fourth dimension and non-Euclidean geometries. In his memoirs, Metzinger would write: "Maurice Princet joined us often... It was as an artist that he conceptualised mathematics, as an aesthetician that he invoked *n*-dimensional continuums. He loved to get the artists interested in the new views on space that had been opened up by Schlegel and some others. He succeeded in that."

poets and critics to meet every Sunday in the studio of the painter Jacques Villon (1875-1963), in the French city of Puteaux. Its members included Metzinger, Gleizes, Juan Gris from Madrid (1887-1927), Le Fauconnier, Léger, Delaunay, the three brothers Jacques Villon, Raymond Duchamp-Villon (1876-1918) and Marcel Duchamp (1887-1968), the Hispano-Cuban Frenchman Francis Picabia (1879-1953), the Czech theosophist Frantisek Kupka (1871-1957) and Apollinaire.

One of the common interests shared by the members of the group was geometry. If we consider the work of these cubists, each of whom has his own characteristic style, we will discover that it was highly geometrical (the fourth dimension, basic geometric shapes, the golden ratio, etc.). One frequent attendee of the meetings of the Puteaux Group was the mathematician Princet, who spoke to them about geometry, particularly the fourth dimension and non-Euclidean geometries. It was through him that they came to be aware of the world of Henri Poincaré and

SECTION D'OR

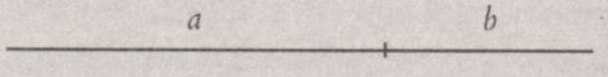
The golden section, golden ratio, or divine proportion, is a geometrical ratio that is of great interest and importance in the world of culture, particularly art.

The golden ratio goes back to Euclid's *Elements of Geometry* and is defined in the following manner: Given a segment divided into two parts with lengths *a* and *b*, the relationship between their lengths has the golden ratio if it is equal to the ratio between the length of the whole segment and the larger part. Mathematically,

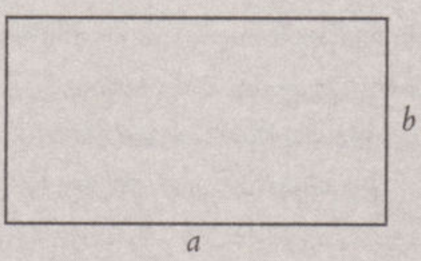
$$\frac{a}{b}=\frac{a+b}{a}.$$

If we call this ratio $\varphi=\frac{a}{b}$, the previous equation becomes the second degree equation $\varphi^2-\varphi-1=0$, whose solution is

$$\varphi=\frac{1+\sqrt{5}}{2}=1.618033...$$



$$\varphi=\frac{a}{b}$$



Esprit Jouffert. One important influence on the relationship to the fourth dimension was the science fiction story *Journey to the Land of the Fourth Dimension* by Gaston de Pawlowski, essential reading among Cubists. A painting by Jean Metzinger, now lost and which was exhibited in 1913, was entitled *Dead Nature (4th Dimension).*

Marcel Duchamp

One of the members of the Puteaux group with the greatest interest in mathematics and the fourth dimension was Marcel Duchamp. His approach differed to that of the other Cubists. While he shared the same concerns as the others, in contrast to them, Duchamp tried to visualise the fourth dimension through his art, and was more mathematical in the way he approached it. Consider an extract from *Conversaciones*

In addition to this, a rectangle with sides *a* and *b* is said to be a golden rectangle if its sides have the golden ratio. In spite of the fact that this appears in relation to Greek and Egyptian rules of beauty, it can only be said that it started to interest not just mathematicians such as Luca Pacioli, but also artists, including Leonardo da Vinci, and other during the Renaissance.

From that point onwards, the golden ratio has been part of culture. Jacques Villon, alongside other Cubists, became interested in it thanks to the French translation of da Vinci's *Trattato della Pittura* in 1910. Their interest in the book resulted in the name Section d'Or, by which the Puteaux Group was also known, even if there were only two members who frequently used the golden ratio in their work, Villon and Gris, although it also appeared in work by Metzinger and Gleizes.

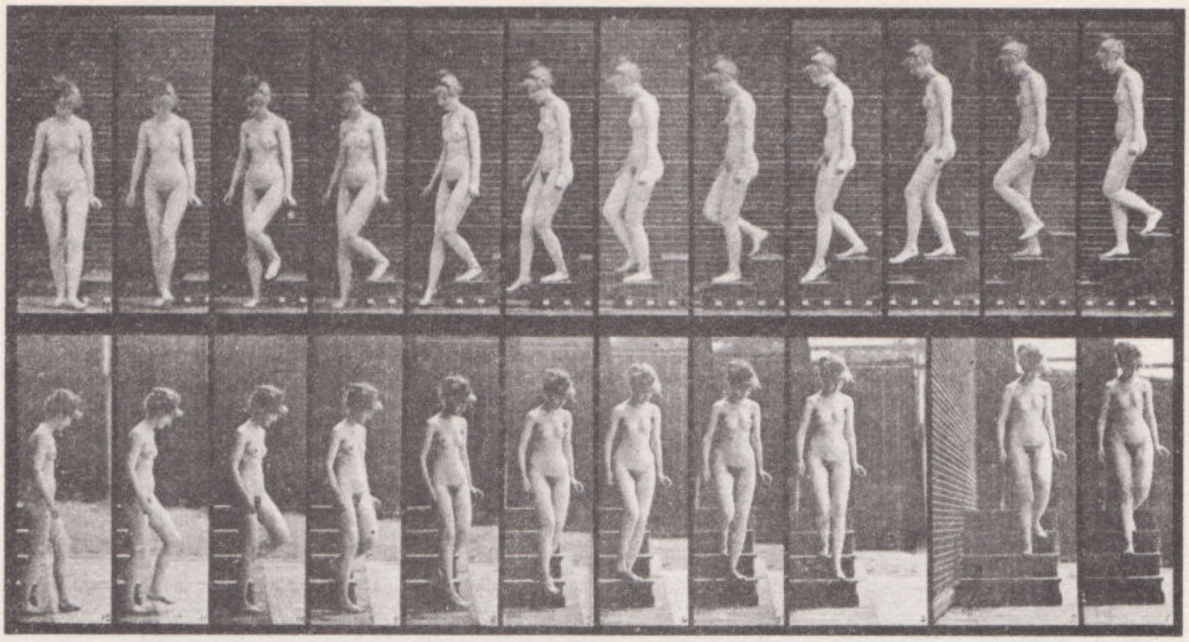
We can appreciate the use of the golden ratio in the work of Juan Gris, in Portrait of Germain Raynal *and* Man in the Café *(above), as well as in* Harlequin Sculpture.

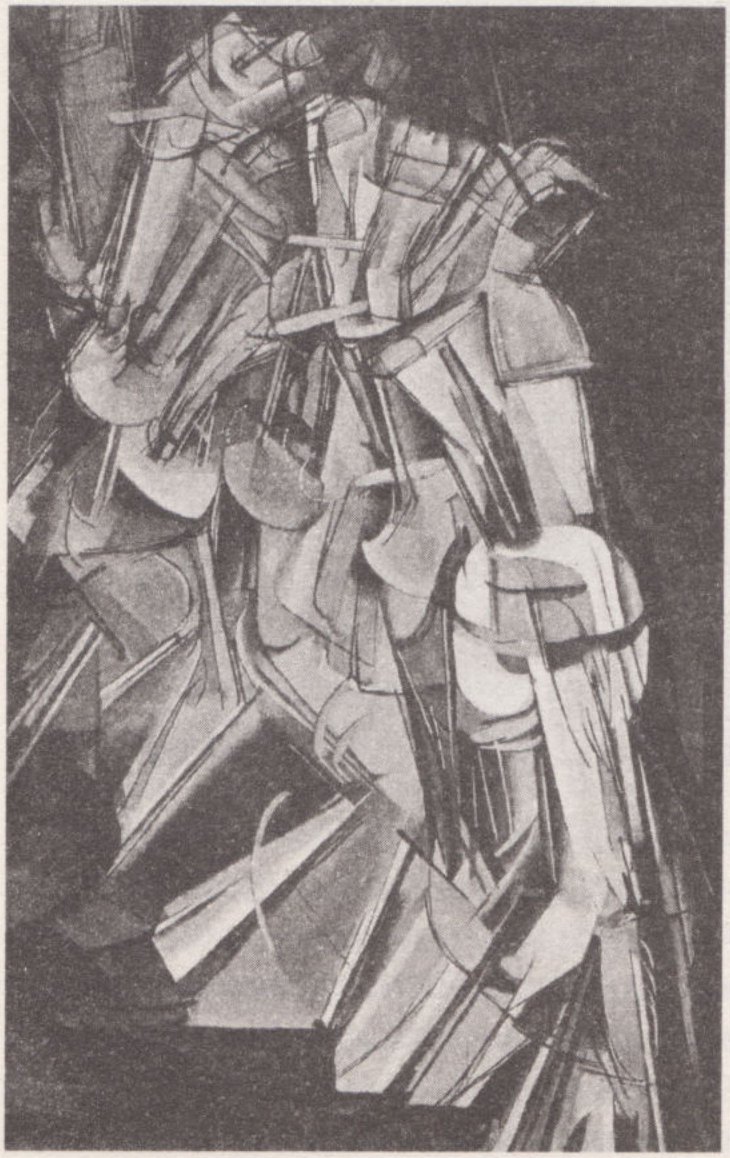
Con Marcel Duchamp (Conversations with Marcel Duchamp) by Cabanne (1966):

> *Cabanne*: Your mathematical knowledge is surprising, and even more so given that you do not have scientific education.
> *Duchamp*: Not entirely. What interested us at that time was the fourth dimension. In *The Green Box* there are copious notes on the fourth dimension. Can you remember a man, I think he was called Povolowski [referring to Pawlowski]? He was an editor from Rue Bonaparte. I forget his name. He wrote articles in a magazine about the popularisation of the fourth dimension, explaining that there were flat beings with only two dimensions, etc. It was really amusing, even during the Cubist period with Princet.
> *Cabanne*: Princet was a false mathematician; he also used irony...
> *Duchamp*: Precisely. We weren't real mathematicians and that's why we believed Princet so much. He gave the impression of knowing a lot."

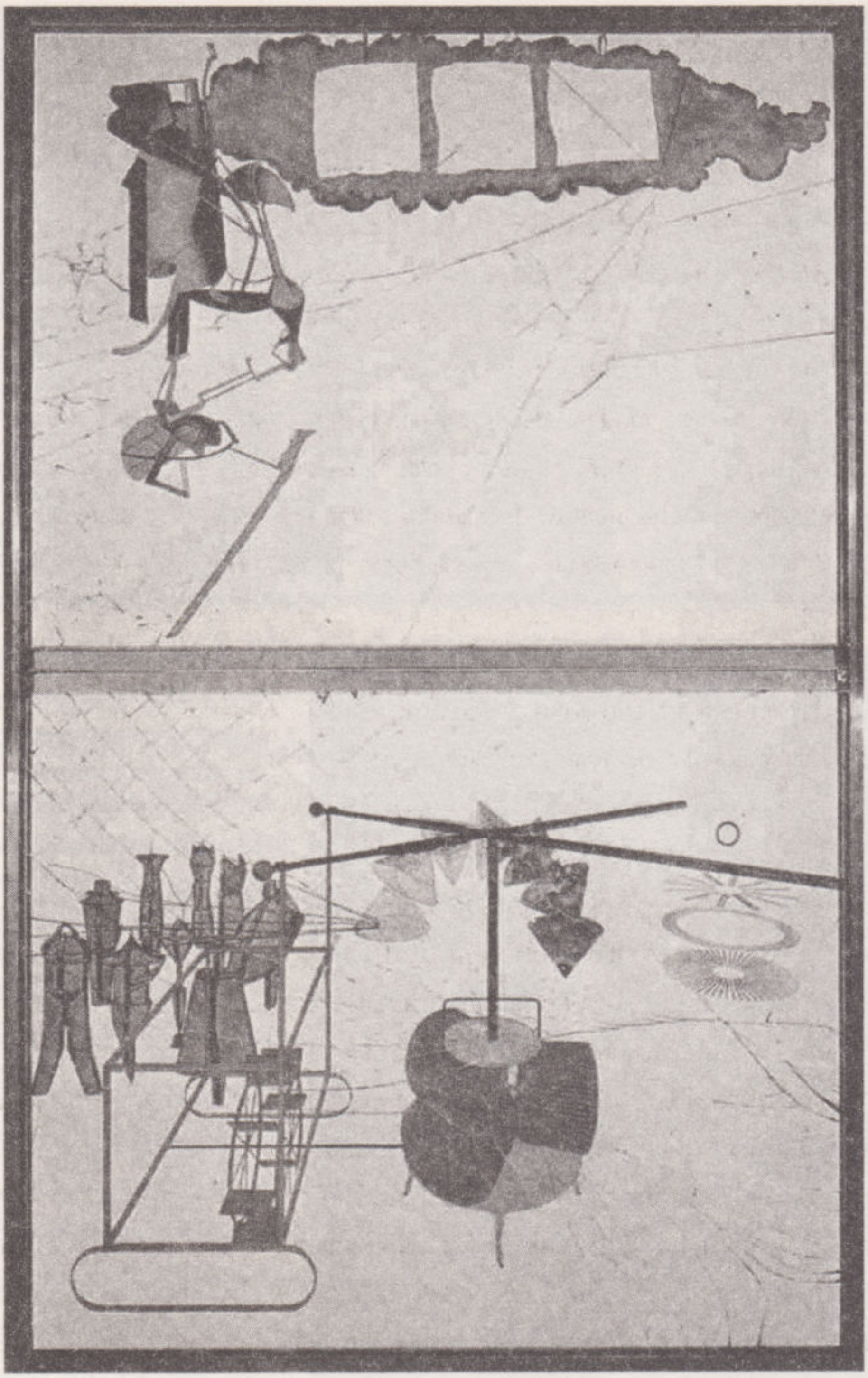
The first of the three works that summarise Marcel Duchamp's interest in the fourth dimension is *Portrait of Chess Players* (1911). From his writing, we know that he had read the work of Jouffret and Poincaré, as well as Gastón de Pawlowski. Jouffret made use of chess as a metaphor for visualising the fourth dimension, identifying it with the mental process of the chess player who played simultaneous matches blind, that is to say, without seeing the boards on which he or she was playing. The theme of this piece is the mental process of the chess player, despite the fact he is not playing simultaneous games. In addition to this, Duchamp, who was a keen chess player throughout his life, claimed in an interview that he had placed its players in an infinite space (as we have already remarked, Cubist theoreticians related the fourth dimension with infinite space).

Subsequently, Duchamp began his study of the static representation of movement, which is one of the techniques for visualising the fourth dimension, that is to say, static and not relative space-time. He talks of what he called elementary parallelism: "...The repetition of a line equivalent to an elementary line (in the same direction at all points) for the generation of a surface. The same parallel used to pass from the plane to volume. The parallel multiplication of a continuous n-dimensional space gives rise to a continuous $(n+1)$-dimensional space." He rediscovered the philosophy of Hinton and others who described the hypercube as a cube moving in a new direction. The work from this period which best reflects this elemental parallelism is *Nude Descending a Staircase No. 2* (1912).





In Nude Descending a Staircase No. 2 *(left), Duchamp mixes cubist ideas and techniques for different perspectives with motion (we could even speak of a fifth dimension), taking his inspiration from a piece by the English photographer Eadweard Muybridge* Woman Walking Downstairs *(above). The work was highly polemic among his Cubist friends, who did not permit movement.*



Marcel Duchamp, The Bride Stripped Bare by Her Bachelors, Even (The Large Glass) *(1915-1923).*

However, Duchamp went even further in his approach to the fourth dimension. At the end of 1912 he made the first notes for the development of his great work, *The Large Glass.* These resulted in *Bride*, a work that Duchamp claimed contained "the first sparkles of the fourth dimension in my work", since the painting represented the three-dimensional projection of the bride who was in the fourth

DUCHAMP AND ANTI-ART

Perhaps the two painters who have had the greatest influence on 20th-century art have been Pablo Picasso and Marcel Duchamp. The first was a painter in the most classic sense of the word, without ever having stopped developing ideas and painting them. Duchamp, however, was an artist who reflected on art, questioning it and even rejecting it; he was essentially a philosopher of art, but from within. One surprising aspect of his career is his artistic inactivity, his anti-art in which Duchamp substitutes painting for reflection upon it. He is reported to have retired from the art world in 1912 and started working in a book shop, which he soon left behind to move to New York. However he continued to work for another ten years, reflecting and painting for his great work *The Bride Stripped Bare by Her Bachelors, Even (The Large Glass)* (1915-1923). As a result of this reflection, he wrote *The White Box* and *The Green Box*, which brought together his ideas for *The Large Glass*, and which contain a number of reflections on the fourth dimension. From then on, he became the anti-artist: He did not make 'works of art' but anti-art, and expressions of such anti-art. Nevertheless he went on to become the most influential figure in 20th-century art.

dimension and would later appear in *The Large Glass.* During those years he continued to work on the issues that interested him, including the fourth dimension, machines, anti-art, psychology and the alienation of man. Upon moving to New York in 1915, he began work on *The Bride Stripped Bare by Her Bachelors, Even (The Large Glass).* In terms of the fourth dimension, the central idea of this period was projection, the visualisation of the fourth dimension and its objects as projections in three dimensional space. In his writings on this work, *The White Box* and *The Green Box*, we can find reflections such as the following: "The shadow generated by a four-dimensional figure in our space is a three-dimensional shadow." Or rather, "…if a shadow is a two-dimensional projection of the three-dimensional world, then the three-dimensional world as we know it is the projection of a four-dimensional universe."

His work *The Bride Stripped Bare by Her Bachelors, Even (The Large Glass),* is just that, a large glass divided into two parts. In the upper part there is the projection of the bride in three-dimensional space, a being who inhabits a four-dimensional universe, whereas in the lower part, there are the nine bachelors in their three-dimensional Euclidean world. The use of glass is, if we refer to his notes, a metaphor for projection: "Using transparent glass and a mirror for perspective." The work has multiple meanings and although great effort has been dedicated to studying it, for many years, the importance of the fourth spatial dimension was overlooked, in much the same way as it has been in all the art from the early 20th century, in spite of direct references by the artists and critics of the time. However, in the 1970s, art historians rediscovered the link between Cubism and the fourth dimension.

THE FINAL CONCLUSION OF *THE LARGE GLASS*

This piece is normally dated between 1915, the year in which Duchamp began work on it after having arrived from New York, and 1923, when he returned to Europe. However the artist felt that he had left it incomplete. After having been shown at the Brooklyn Museum in 1926, the people taking it down put the two pieces of glass, the part with the bride and the one with her bachelors, on top of the other. While being transported in the lorry, the glass broke as it bumped along the poor road. Finally, in 1936, Duchamp restored the painting with the help of a glazier, and it was then, upon seeing the symmetrical layout of the cracks in the two parts of the piece, with the bride finally in communication with the bachelors, that he considered the piece to be complete.

The fourth dimension in different movements of 20th-century art

Throughout the 20th century, the fourth spatial dimension was not always present in art. It is possible to distinguish four stages:

1900-1930: The golden era (essentially up to 1920).

1920-1950: The theory of relativity comes into play and, in 1936, artists sign the Dimensionist Manifesto, which is open to space but fundamentally relativist. Only Dadaism and Surrealism continued to be interested in the fourth spatial dimension.

1950-1970: Abandonment of the theme of the fourth spatial dimension, except for notable artists such as Salvador Dalí and the American Irene Rice Pereira (1902-1971).

From the 1970s: Re-emergence of interest in the fourth spatial dimension. There are currently many artists and movements with a renewed interest in the material discussed in this book.

As we have remarked, the fourth dimension became a symbol of liberation for artists, a source of ideas, a new language and a new concept of space. The belief in the fourth dimension gave them the strength to distance themselves from visual reality and completely break with the single vanishing point perspective which represented the three-dimensional Euclidean world. In addition to this, interest in the fourth dimension represented an alternative approach to the idealist visions of a higher reality underpinning the birth of abstract art. There is a great fascination with the fourth dimension, although each movement and each artist relates it to different aspects: the break with perspective, motion, colour, time, infinite space, memory, gravity, anti-gravity, mysticism...

In addition to Cubism, the 20th-century art movements with an interest in the fourth spatial dimension, including the space-time continuum, include Italian and Russian Futurism, Suprematism, Constructivism, American Modernism, the De Stijl Movement, Surrealism and Dadaism. We highlight some of these below, alongside their most representative artists.

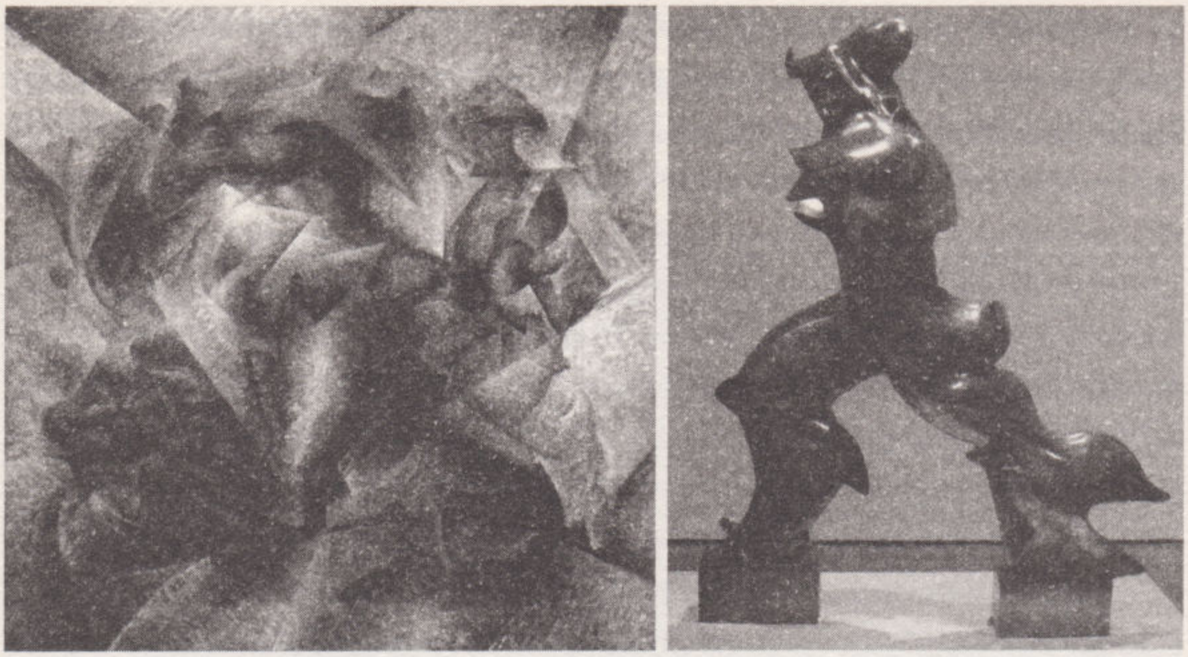
ABSTRACTION IN 20TH-CENTURY ART

A process of abstraction took place in 20th-century art. Artists no longer wished to make a copy of an object, nor of a real essence – this could be done by a photograph – and they wanted to break with the interpretation of art as realist painting that chronicled history. What mattered was not the object being painted, but painting itself. They wished to see the world from different points of view, not painting what they saw, but instead a higher reality, which could also be the inner world. They questioned truth, the reality of things, the world, and attempted to work with great universal and utopian ideas.

Twentieth-century art followed three fundamental trends: expression (related to feelings), fantasy (labyrinths of the mind) and geometric abstraction (emphasising order and formal structure). All artistic movements from the past century are a mix of these trends.

Futurism

The Futurist movement created a dynamic vision of the fourth dimension. Its adherents rejected the static vision adopted by the Cubists. For them the new dimension was motion, the space-time continuum. The Italian Futurist Umberto



Two works by Umberto Boccioni, Dynamism of a Cyclist *(left) and* Unique Forms of Continuity in Space.

Boccioni (1882-1916), one of the theoreticians and co-authors behind the Futurist manifestos writes: "*Dynamic form* is a species of fourth dimension, both in painting and sculpture. With the unique form that gives continuity in space we created a form that is the sum of the potential unfolding of the three known dimensions. Spiritualisation will be given by mathematical values; by geometric dimensions."

Boccioni was profoundly influenced by Hinton's hyperdimensional philosophy, or its Italian equivalents, and the use of motion in order to understand the fourth dimension. His works, alongside those of the other Futurists are spatial projections of the space-time continuum between two specific moments.

One of Boccioni's most characteristic pieces, featured on the Italian 20 Euro cent coin is the sculpture *Unique Forms of Continuity in Space*, which represents the movement of a person walking. It is the spatial projection, in this three-dimensional case (since it is a sculpture) of the rail track of this person in space-time.

Similarly, the paintings of the Futurists are the flat spatial projections of the space-time continuum, of motion. Examples of this include *Dynamism of a Football Player* and *Dynamism of a Cyclist*, by Boccioni; *Blue Dancer*, by Gino Severini (1883-1966); *Dynamism of a Dog on a Leash* (1912) and *Flight of the Swallows* (1913), by Giacomo Balla (1871-1958).

The fourth dimension was also a topic of interest for Russian Futurism. Their art was highly influenced by the arrival of the Parisian avant-gardes, as well as the four-dimensional philosophy of the theosophist, Piotr D. Ouspensky. Its members include artists such as Mikhail Larionov (1881-1964), Natalia Goncharova (1881-1962), Mikhail Matyushin (1861-1934) and Kazimir Malévich (1878-1935). The last three of these figures also made up part of what is known as Cubo-Futurism, a mixture of Cubism and Futurism. Malévich's piece *The Knife Grinder* is typical of this phase.

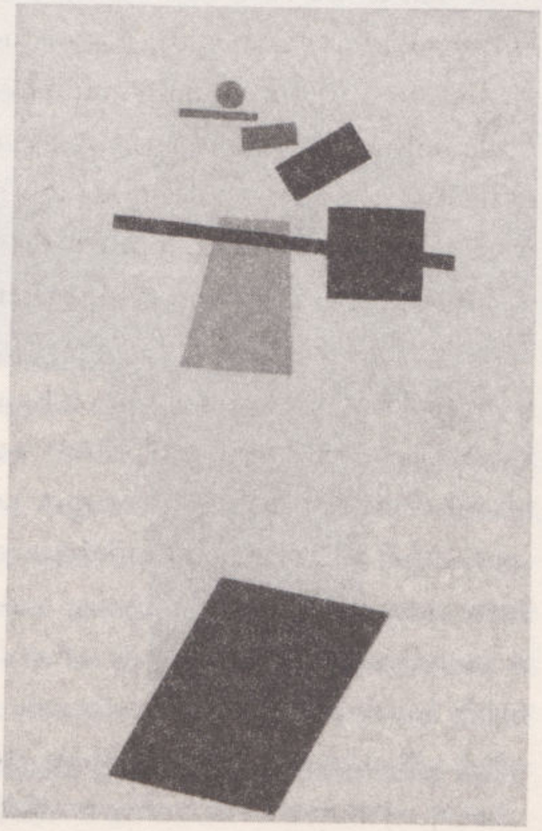
Suprematism

Developed in Russia by Malévich, Suprematism focused on basic geometric shapes, specifically the circle and the square. The movement sought pure feeling, trying to shock people's complacent perception of the world to lead them to a sort of cosmic consciousness, pure feeling. This is created as a result of the tensions and movements between colour, shape and the base of the work.

Similar to Russian futurism, Suprematism was related to Hinton and Ouspensky's hyperspatial philosophy. One movement extremely close to the latter was Kandinsky's *Der Blaue Reiter*, also heavily influenced by Ouspensky's work. As such, the existence of a spiritual component in the Suprematist approach to the fourth dimension should not come as a surprise. Additionally, Malévich's work from this period recalls the flat sections of shapes from higher dimensions.

Linda D. Henderson also explains to us that "in addition to geometry and motion, there is a third way in which hyperspatial philosophy could have brought Malévich to proclaim the presence of the fourth dimension in certain Suprematist works related to the new space, free from gravity."

At the first showing of Malévich's Suprematist work, the 0.10 Exhibition (1915), the last exhibition of Futurist paintings, of



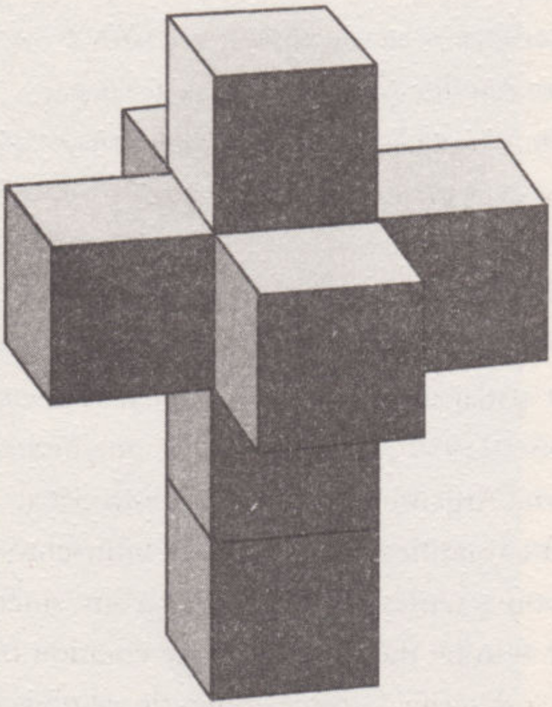
Kazimir Malévich, Suprematism Painter Like Realism of a Football Player – Colour Masses of the Fourth Dimension *(1915).*

the 39 works presented, five had subtitles related to the fourth dimension, such as, *Painterly Realism of a Football Player – Colour Masses in the Fourth Dimension* and *Painterly Realism, Boy with Knapsack – Colour Masses in the Fourth Dimension*, known as *Red Square and Black Square.*

Surrealism

Surrealism is an artistic movement which provoked interest both on account of its geometric or spatial component of the fourth dimension (including static space-time), and its spiritual one. The Catalan Salvador Dalí (1904-1989), who had a keen interest in science, especially mathematics, towers above the other figures of this movement. In his work, we can find references to the golden ratio, catastrophe theory, fractals, topology, geometric forms, stereoscopy, projective geometry and optical illusions, as well as higher dimension spaces.

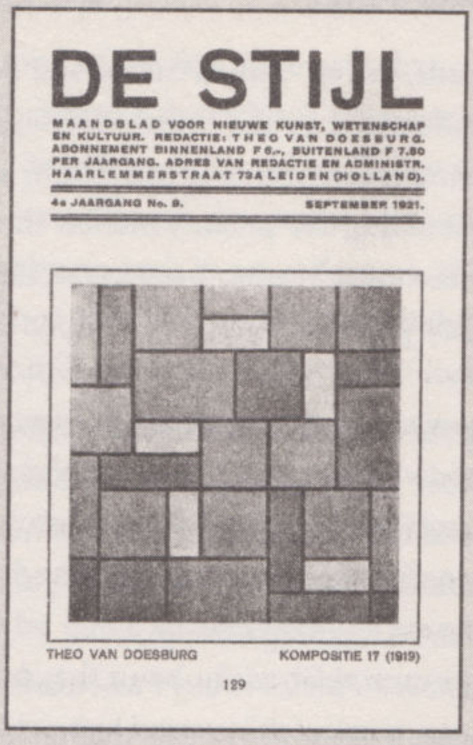
Two of Dalí's works stand out on account of their relationship with the fourth dimension, *Crucifixion (Corpus Hypercubus)* and *In Search of the Fourth Dimension*. In the first of these, we find the unfolding of the hypercube (see the previous chapter), which is a three-dimensional cross on which Jesus is being crucified, a reference to the religious connotations of the fourth dimension. The cross, the unfolding of the hypercube, becomes a symbol of passing from our world (three-dimensional space) to heaven (in the fourth dimension). In the second we find many of the elements that have been discussed in this text (polyhedral forms, the myth of Plato's cave, religion, projections, etc.) as well as the theory of relativity.



Cross similar to the one imagined by Salvador Dalí on which Jesus Christ is being crucified.

The writings of Óscar Domínguez (1906-1957) would appear to be the most scientific within the interest that the fourth dimension aroused among the Surrealists. There are two types of works related to higher dimensions. On

DE STIJL



DE STIJL

MAANDBLAD VOOR NIEUWE KUNST, WETENSCHAP EN KULTUUR. REDACTIE: THEO VAN DOESBURG. ABONNEMENT BINNENLAND F 6.–, BUITENLAND F 7.50 PER JAARGANG. ADRES VAN REDACTIE EN ADMINISTR. HAARLEMMERSTRAAT 73A LEIDEN (HOLLAND).

4e JAARGANG No. 9. SEPTEMBER 1921.

THEO VAN DOESBURG KOMPOSITIE 17 (1919)

129

Cover of the journal De Stijl *from 1921, which shows a painting by Theo van Doesburg,* Composition XVII *(1919).*

De Stijl (The Style) was a movement and journal founded in Holland in 1917 by the painter and architect Theo van Doesburg (1883-1931), and which counted Piet Mondrian (1872-1944) as a member. It was rooted in Analytic Cubism but evolved into pure geometric abstraction. We can discover the artists' concerns about the fourth dimension by reading the articles in the journal, which include, for example, Severini's *The Measurement of Space and the Fourth Dimension*, fragments of Poincaré's *Science and Hypothesis* and van Doesburg and Mondrian's reflections on the subject. According to the former, in expressing the spiritual, in making spirit an artefact, man will be forced to a moto-stereometric form of expression. This moto-stereometric form of expression represents the appearance of a four dimensional world in a world of three dimensions. For Mondrian, The new trends have been ascribed to a great understanding of the fourth dimension and, in fact, the idea that the fourth dimension is manifested in new art through the total or partial destruction of three-dimensional natural order, and through the construction of a new plasticity in line with a less limited vision.

the one hand, there are the so-called 'highly spatial cosmic paintings', such as *Landscape of Arteries* (1939) and *Nostalgia of Space* (1939), which contain polyhedral shapes. On the other hand, together with the Argentinian writer Ernesto Sábato, towards 1939 he introduced the theory of the solidification of time, or litho-chronism. His 'litho-chronic' surfaces were in some senses related to Futurism, since they represented motion. These surfaces came to be the pictorial representation of the railway of the space-time continuum that describes a three-dimensional object between two moments in time. A representative piece from this period is *The Litho-Chronic Sword or the Celebration of the Bull* (1939).

The fourth dimension in the art of the United States

Although lagging behind the Parisian avant-garde, in the United States, and particularly in New York, the art world was also influenced by the fourth dimension. Like the rest of the world, as well as the geometrical revolution itself, theosophy and science fiction made a great contribution to extending the public's interest in the fourth dimension. Moreover, it should also be noted that Hinton lived in the United States for a number of years and was a highly popular figure who helped disseminate the subject. In the art world, the publications of the theosophist and architect Claude F. Bragdon were significant, influencing Max Weber (who belonged to the Stieglitz Group) upon return from his time in Paris, as was the extended presence of members of the Puteaux Group in New York, specifically Picabia (who was in contact with the Stieglitz Group) and Duchamp (with the Arensberg Group).

Max Weber, author of *The Fourth Dimension from a Plastic Point of View*, made use of the fourth dimension in his art and there is even a piece with a title that includes the term: *Interior of the Fourth Dimension* (1913). Weber's fourth dimension was a mix of the Cubist vision to which he added sequential forms, very much in the style of Duchamp's *Nude Descending a Staircase No. 2*. For Weber the fourth dimension also had a spiritual and religious element.

The American artist Max Weber, born in Białystok, a Polish city occupied by Russia, in a photograph in 1914.

Epilogue

In spite of the fact that more than a century has passed since the fourth dimension first captured the public's attention with such intensity and interest, it continues to be a hugely topical subject. To see this, it is sufficient to go onto the Web and type the term "fourth dimension" into any search engine. A great number of hits related to subjects such as art, religion, science fiction, popular science, applications in the field of visualisation (e.g. in health), technology will appear, in addition to purely scientific ones. Moreover, many works on the subject continue to be published to this very day, including fiction, essays and popular science. There is even literature for children and young adults which is related to the fourth dimension, and the term crops up naturally in television series. But beyond this, if a conference is given on the fourth dimension nowadays, or the topic comes up at a social event, it will still be quick to captivate the attention of those present, and the same issues debated at the end of the 19th century will be, in many respects, relived.

But this renewed interest is not only confined to society. With string theory, which discusses models that describe how our universe could exist within a space of higher dimensions (10, 11 or 26), both spatial and temporal, science itself has once again made the fourth dimension, or rather higher spatial dimensions, fashionable. Moreover, the media has given extensive coverage to Europe's modern particle accelerator, the LHC (Large Hadron Collider), which will make it possible to show what once seemed impossible: the existence of extra spatial dimensions, even ones beyond the fourth.

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The Fourth Dimension

Is our Universe the shadow of another?

The idea that there are extra dimensions beyond our senses has fascinated scientists over the centuries. Although it may seem impossible to imagine a world with more than three dimensions, mathematics shows us how we can conceive of multi-dimensional space, and opens our eyes to a world of ghostly apparitions and impossible objects.